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Generalized Euler angle parametrization for $SU(N)$

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Abstract

In a previous paper [1] an Euler angle parametrization for $SU(4)$ was given. Here we present the derivation of a generalized Euler angle parametrization for $SU(N)$. The formula for the calculation of the Haar measure for $SU(N)$ as well as its relation to Marinov's volume formula for $SU(N)$ [2, 3] will also be derived. As an example of this parametrization's usefulness, the density matrix parametrization and invariant volume element for a qubit/qutrit, three qubit and two three-state systems, also known as two qutrit systems [4], will also be given.

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1. Introduction

The importance of group theory in understanding quantum mechanical processes has grown over the past 100 years as more and more physicists recognized that

> The basic principles of quantum mechanics seem to require the postulation of a Lie algebra of observables and a representation of this algebra by skew-Hermitian operators on a Hilbert space [5].

Today one finds applications of group theory in numerous research areas, from high-energy particle theory to experimental nano-scale physics. Any physical system that exhibits symmetries, or that can be thought of as Hermann indicated above, will have a group associated with it. In this paper we are mostly concerned with representations of, and applications on, $SU(N)$. This group is shown in numerous areas of study, most notably in numerical calculations concerning entanglement and other quantum information parameters. In order to assist in these numerical calculations, we have produced a parametrization of $SU(N)$ that should offer some computational benefit.

We will begin this paper by deriving a general Euler angle parametrization for $SU(N)$. Afterward, a general equation for the differential volume element, otherwise known as the Haar measure, for $SU(N)$ will be derived. Then we will show that this parametrization yields the familiar invariant volume element, generated from the integration of the Haar measure, for $SU(N)$ as derived by Marinov [2, 3]. Finally, the parametrization of N by N density matrices with regards to the general Euler angle parametrization for $SU(N)$ will be shown. As an illustration of the usefulness of the parametrization, representations of qubit/qutrit, three qubit and two qutrit states will be given.

2. Review: Euler angle parametrization from $SU(2)$ to $SU(4)$

For a $U \in SU(2)$ the Euler angle representation can be found in any good textbook on quantum mechanics or Lie algebras (see, for example, [6–8])

$$U = \mathrm{e}^{\mathrm{i}\sigma_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\sigma_2\alpha_2}\,\mathrm{e}^{\mathrm{i}\sigma_3\alpha_3}. \tag{1}$$

For a $U \in SU(3)$ the Euler angle parametrization was initially given in [9, 10] and later in [11]

$$U = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_5}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_6}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_8}. \tag{2}$$

For a $U \in SU(4)$ the Euler angle parametrization was initially given in [1]

$$\begin{aligned} U = {} & \mathrm{e}^{\mathrm{i}\lambda_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_5}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_6}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_8} \\ & \times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{10}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{11}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{12}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{13}}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{14}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{15}}. \end{aligned} \tag{3}$$

We would like to extend this work for a $U \in SU(N)$.

3. Lie algebra for $SU(N)$

From [6] we already know how to construct the λ_i for arbitrary $SU(N)$.[3]

1. For every $i, j = 1, 2, 3, \ldots, N$; $i < j$, we define two N by N matrices

   $$[\lambda^{\{1\}}(i,j)]_{\mu\nu} = \delta_{j\mu}\delta_{i\nu} + \delta_{j\nu}\delta_{i\mu} \qquad [\lambda^{\{2\}}(i,j)]_{\mu\nu} = -\mathrm{i}(\delta_{i\mu}\delta_{j\nu} - \delta_{i\nu}\delta_{j\mu}) \tag{4}$$

   which form $N(N-1)$ linearly independent matrices.[4]
2. We construct further $N-1$ matrices according to

   $$\lambda_{n^2-1} = \sqrt{\frac{2}{n^2-n}} \begin{pmatrix} (1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1)_{n-1} & 0 & \ldots & 0 \\ 0 & 0 & 0 & -(n-1) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}_{N\times N} \tag{5}$$

   for $n = 2, 3, \ldots, N$.

By following this convention, N^2-1 traceless matrices can be generated. These matrices form a basis for the corresponding vector space and thus a representation of the $SU(N)$ generators [6]. For example, for $N = 2$ we generate the well-known Euler σ matrices ($i < j$, $\{i, j\} \leqslant 2$):

[3] Georgi [12] also gives a method for constructing the $N^2 - 1$ λ_i matrices for $SU(N)$.
[4] We follow the standard physics practice by using Hermitian generators.

$$\sigma_1 = [\lambda^{\{1\}}(1,2)]_{\mu\nu} = \delta_{2\mu}\delta_{1\nu} + \delta_{2\nu}\delta_{1\mu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6}$$

$$\sigma_2 = [\lambda^{\{2\}}(1,2)]_{\mu\nu} = -\mathrm{i}(\delta_{1\mu}\delta_{2\nu} - \delta_{1\nu}\delta_{2\mu}) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{7}$$

and

$$\sigma_3 = \lambda_{2^2-1} = \lambda_3 = \sqrt{\frac{2}{2^2-2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

4. Deriving the Euler angle parametrization for $SU(N)$

By following the work of Biedenharn [13] and Hermann [5] we can now generate a Cartan decomposition of $SU(N)$ for $N > 2$.[5] First, we look at the N by N, Hermitian, traceless, λ_i matrices formulated in the previous section. This set is linearly independent and is the lowest dimensional faithful representation of the $SU(N)$ Lie algebra. From these matrices we can then calculate their commutation relations

$$[\lambda_i, \lambda_j] = 2\mathrm{i} f_{ijk}\lambda_k \qquad f_{ijk} = \frac{1}{4\mathrm{i}}\mathrm{Tr}[[\lambda_i, \lambda_j]\lambda_k] \tag{9}$$

and by observation of the corresponding structure constants f_{ijk} one can see the relationship in the algebra that can help generate the Cartan decomposition of $SU(N)$ (shown for $SU(3)$ in [9] and for $SU(4)$ in [1]).

Knowledge of the structure constants allows us to define two subspaces of the $SU(N)$ group manifold hereafter known as K and P. From these subspaces, there correspond two subsets of the Lie algebra of $SU(N)$, $L(K)$ and $L(P)$, such that for $k_1, k_2 \in L(K)$ and $p_1, p_2 \in L(P)$,

$$[k_1, k_2] \in L(K) \qquad [p_1, p_2] \in L(K) \qquad [k_1, p_2] \in L(P). \tag{10}$$

For $SU(N)$, $L(K) = \{\lambda_1, \ldots, \lambda_{(N-1)^2-1}, \lambda_{N^2-1}\}$ and $L(P) = \{\lambda_{(N-1)^2}, \ldots, \lambda_{N^2-2}\}$.[6] Given that we can decompose the $SU(N)$ algebra into a semi-direct sum [14]

$$L(SU(N)) = L(K) \oplus L(P) \tag{11}$$

we therefore have a decomposition of the group,

$$U = K \cdot P. \tag{12}$$

From [8] we know that $L(K)$ comprises the generators of the $SU(N-1)$ subalgebra of $SU(N)$, thus K will be the $U(N-1)$ subgroup obtained by exponentiating this subalgebra, $\{\lambda_1, \ldots, \lambda_{(N-1)^2-1}\}$, combined with λ_{N^2-1} and thus can be written as (see [1, 9] for examples)

$$K = [SU(N-1)] \cdot \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\phi} \tag{13}$$

[5] The reason for this will become apparent in the following discussion.

[6] For example, for $SU(4)$ we have $L(K) = \{\lambda_1, \lambda_2, \ldots, \lambda_8, \lambda_{15}\}$ and $L(P) = \{\lambda_9, \lambda_{10}, \ldots, \lambda_{14}\}$ [1]. By definition, $L(K)$ and $L(P)$ do not have any elements in common. This means that for $N = 2$, the second λ_1 element that is generated by this construction as an element of $L(P)$ must be discarded. Similarly, the undefined λ_0 element in $L(K)$ must also be discarded. For $N \geqslant 3$ one does not generate any duplications.

where $[SU(N-1)]$ represents the $(N-1)^2-1$ term Euler angle representation of the $SU(N-1)$ subgroup.

Now, as for P, of the $2(N-1)$ elements in $L(P)$ we choose the λ_2 analogue, $\lambda_{X_{SU(N)}}$, for $SU(N)$ and write any element of P as

$$P = K' \cdot \mathrm{e}^{\mathrm{i}\lambda_{X_{SU(N)}}\psi} \cdot K'' \tag{14}$$

where K' and K'' are copies of K and $\lambda_{X_{SU(N)}}$ is given by the N by N matrix

$$\lambda_{X_{SU(N)}} = \lambda_{(N-1)^2+1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -\mathrm{i} \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ i & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{N\times N}. \tag{15}$$

Unfortunately, at this point in our derivation, we have overparametrized U by $2N^2-6N+5$ elements

$$U = K \cdot K' \cdot \mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2+1}\psi} \cdot K''. \tag{16}$$

But, if we recall that U is a product of operators in $SU(N)$, we can 'remove the redundancies', i.e. the first K' component as well as the $N-1$ Cartan subalgebra elements of $SU(N)$ in the original K component, to arrive at the following product (again, see [1, 9, 10] for examples):

$$\begin{aligned} U &= [SU(N-1)] \cdot \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\phi}\, \mathrm{e}^{-\mathrm{i}\lambda_{N^2-1}\phi}\, \mathrm{e}^{-\mathrm{i}\lambda_{(N-1)^2-1}\phi_1}\, \mathrm{e}^{-\mathrm{i}\lambda_{(N-2)^2-1}\phi_2} \cdots \mathrm{e}^{-\mathrm{i}\lambda_3\phi_{N-1}} \\ &\quad \times \mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2+1}\psi} \cdot [SU(N-1)] \cdot \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\phi} \\ &= [SU(N-1)] \cdot \mathrm{e}^{-\mathrm{i}\lambda_{(N-1)^2-1}\phi_1} \cdots \mathrm{e}^{-\mathrm{i}\lambda_3\phi_{N-1}}\, \mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2+1}\psi} \cdot [SU(N-1)] \cdot \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\phi}. \end{aligned} \tag{17}$$

By insisting that our parametrization must truthfully reproduce known vector and tensor transformations under $SU(N)$, we can remove the last 'redundancy', the final N^2-5N+5 elements in K,[7] and, after rewriting the parameters, get

$$U = \left(\prod_{2\leqslant k\leqslant N} A(k)\right) \cdot [SU(N-1)] \cdot \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\alpha_{N^2-1}} \qquad A(k) = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)}}\, \mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)}}. \tag{18}$$

This equation, effectively a recurrence relation for the Euler angle decomposition of $SU(N)$, can be further rewritten in a more explicit, and therefore final, form

$$\begin{aligned} U &= \left(\prod_{2\leqslant k\leqslant N} A(k, j(N))\right) \cdot \left(\prod_{2\leqslant k\leqslant N-1} A(k, j((N-1))\right) \cdots (A(2, j(2))) \\ &\quad \times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{N^2-(N-1)}} \cdots \mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2-1}\alpha_{N^2-2}}\, \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\alpha_{N^2-1}} \\ &= \prod_{N\geqslant m\geqslant 2}\left(\prod_{2\leqslant k\leqslant m} A(k, j(m))\right) \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{N^2-(N-1)}} \cdots \mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2-1}\alpha_{N^2-2}}\, \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\alpha_{N^2-1}} \end{aligned} \tag{19}$$

$$A(k, j(m)) = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}}\, \mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}$$

$$j(m) = \begin{cases} 0 & m = N \\ \sum_{0\leqslant l\leqslant N-m-1} 2(m+l) & m \neq N. \end{cases}$$

[7] For $N = 3$, there are no redundancies. In fact, the -1, $(3^2-5*3-5=-1)$, that occurs here is a result of removing one too many Cartan subalgebra elements from the end of K in the previous step. For $N = 3$, one must *restore* a Cartan subalgebra element, in this case $\mathrm{e}^{\mathrm{i}\lambda_3\alpha_4}$, back into its original position in the K component. For $N > 3$ this situation does not occur.

In this form, the important λ_i matrices for equation (19) are

$$\lambda_3 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}_{N\times N}$$

$$\lambda_{(k-1)^2+1} = \begin{pmatrix} \begin{bmatrix} 0 & \dots & -\mathrm{i} \\ \dots & \dots & \dots \\ \mathrm{i} & \dots & 0 \end{bmatrix}_{k\times k} & \dots & 0 \\ 0 & \dots & 0 \end{pmatrix}_{N\times N} \qquad k < N$$

$$\lambda_{(N-1)^2+1} = \begin{pmatrix} 0 & \dots & -\mathrm{i} \\ \dots & \dots & \dots \\ \mathrm{i} & \dots & 0 \end{pmatrix}_{N\times N} \qquad k = N \tag{20}$$

$$\lambda_{N^2-1} = \sqrt{\frac{2}{N^2-N}} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -(N-1) \end{pmatrix}_{N\times N}.$$

Note that even though we restricted ourselves to $N > 2$ for the Cartan decomposition, equation (19) is valid for $N \geqslant 2$. For our purposes it is enough to note that this parametrization is special unitary by construction and can be shown to cover the group by modifying the ranges that follow and substituting them into the parametrization of the characters [11].[8]

5. Procedure for calculating the Haar measure and group volume for $SU(N)$

Taking the Euler angle parametrization given by equation (19) we now develop the differential volume element, also known as the Haar measure, for the group. We proceed by using the method originally given in [15] and developed for $SU(3)$ in [9, 10] and for $SU(4)$ in [1]; take a generic $U \in SU(N)$ and find the matrix

$$U^{-1} \cdot \mathrm{d}U \tag{21}$$

of left invariant 1-forms, then take the determinant of the matrix of their expansion coefficient[9].

To begin, we take the transpose of U of generate

$$\begin{aligned} u &= U^T \\ &= \mathrm{e}^{\mathrm{i}\lambda^T_{N^2-1}\alpha_{N^2-1}} \cdot [SU(N-1)]^T \cdot \left(\prod_{N\geqslant k\geqslant 2} A(k)^T\right) \\ &= \mathrm{e}^{\mathrm{i}\lambda^T_{N^2-1}\alpha_{N^2-1}}\, \mathrm{e}^{\mathrm{i}\lambda^T_{(N-1)^2-1}\alpha_{N^2-2}} \cdots \mathrm{e}^{\mathrm{i}\lambda^T_3\alpha_{N^2-(N-1)}} \prod_{2\leqslant m\leqslant N} \left(\prod_{m\geqslant k\geqslant 2} A(k, j(m))^T\right) \end{aligned} \tag{22}$$

[8] One may be somewhat distressed by our 'removal of the redundancies' statement that precluded the development of equation (19), so we offer another, geometrical, argument for the form of the parametrization in appendix A.

[9] A detailed explanation of this method can be found in [1]. We should note that one can take the determinant of the matrix of the expansion coefficients of the $N^2 - 1$ right invariant 1-forms which also yields the Haar measure in question. This is due to the fact that a compact simply-connected real Lie group has a bi-invariant measure, unique up to a constant factor. Such a group is usually referred to as 'unimodular' [8].

where

$$A(k, j(m))^T = \mathrm{e}^{\mathrm{i}\lambda^T_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}\,\mathrm{e}^{\mathrm{i}\lambda^T_3\alpha_{(2k-3)+j(m)}}$$
$$j(m) = \begin{cases} 0 & m = N \\ \sum_{0\leqslant l\leqslant N-m-1} 2(m+l) & m \neq N. \end{cases} \qquad (23)$$

An observation of the components of our Lie algebra sub-set

$$\{\lambda_{(k-1)^2+1}, \lambda_{k^2-1}\} \qquad 2 \leqslant k \leqslant N \qquad (24)$$

shows that the transpose operation is equivalent to making the following substitutions:

$$\lambda^T_{(k-1)^2+1} \to -\lambda_{(k-1)^2+1} \qquad \lambda^T_{k^2-1} \to \lambda_{k^2-1} \qquad (25)$$

for $2 \leqslant k \leqslant N$ in equation (22) generating

$$u = \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\alpha_{N^2-1}}\,\mathrm{e}^{\mathrm{i}\lambda_{(N-1)^2-1}\alpha_{N^2-2}}\cdots\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{N^2-(N-1)}}\prod_{2\leqslant m\leqslant N}\left(\prod_{m\geqslant k\geqslant 2} A(k, j(m))'\right)$$
$$A(k, j(m))' = \mathrm{e}^{-\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}} \qquad (26)$$
$$j(m) = \begin{cases} 0 & m = N \\ \sum_{0\leqslant l\leqslant N-m-1} 2(m+l) & m \neq N. \end{cases}$$

With this form in hand, we then take the partial derivative of u with respect to each of the $N^2 - 1$ parameters. In general, the differentiation will have the form

$$\frac{\partial u}{\partial\alpha_l} = E^L C(\alpha_l) E^{-L} u \qquad (27)$$

where we have used a 'shorthand' representation with

$$C(\alpha_l) \in \mathrm{i} * \{-\lambda_{(k-1)^2+1}, \lambda_{k^2-1}\} \qquad 2 \leqslant k \leqslant N \qquad (28)$$

and

$$E^L = \mathrm{e}^{C(\alpha_{N^2-1})\alpha_{N^2-1}}\cdots\mathrm{e}^{C(\alpha_{l+1})\alpha_{l+1}} \qquad E^{-L} = \mathrm{e}^{-C(\alpha_{l+1})\alpha_{l+1}}\cdots\mathrm{e}^{-C(\alpha_{N^2-1})\alpha_{N^2-1}}. \qquad (29)$$

By using these equations and the Baker–Campbell–Hausdorff relation,

$$\mathrm{e}^X Y \mathrm{e}^{-X} = Y + [X, Y] + \tfrac{1}{2}[X, [X, Y]] + \cdots \qquad (30)$$

we are able to consecutively solve equation (27) for $l = \{N^2 - 1, \ldots, 1\}$, leading to a set of N by N matrices which can be expanded in terms of the $N^2 - 1$ *transposed* elements of the $SU(N)$ Lie algebra with expansion coefficients c_{lj} given by trigonometric functions of the group parameters α_i:

$$M_l \equiv \frac{\partial u}{\partial\alpha_l}u^{-1} = E^L C(\alpha_l) E^{-L} = \sum_{N^2-1\geqslant j\geqslant 1} c_{lj}\lambda^T_j. \qquad (31)$$

The λ^T_j can be generated by using the methods contained in section 3.

Now the coefficients c_{lj} are the elements of the determinant in question. They are found by evaluating the following trace [6]:

$$c_{lj} = \frac{-\mathrm{i}}{2}\,\mathrm{Tr}\left[\lambda^T_j \cdot M_l\right]. \qquad (32)$$

The index l corresponds to the specific α parameter, the j corresponds to the specific *transposed* element of the algebra. Both the l and j indices run from $N^2 - 1$ to 1. The determinant of this $N^2 - 1$ by $N^2 - 1$ matrix yields the differential volume element, also known as the Haar measure for the group, $\mathrm{d}V_{SU(N)}$ that, when integrated over the correct values for the ranges of the parameters and multiplied by a derivable normalization constant, yields the volume for the group.

The full N^2-1 by N^2-1 determinant $\mathrm{Det}[c_{lj}]$, $l, j \in \{N^2-1, \ldots, 1\}$, can be done, or one can note that the matrix can be written as

$$C_{SU(N)} = \begin{Vmatrix} c_{N^2-1,N^2-2} & c_{N^2-1,N^2-3} & \cdots & c_{N^2-1,1} & c_{N^2-1,N^2-1} \\ c_{N^2-2,N^2-2} & c_{N^2-2,N^2-3} & \cdots & c_{N^2-1,1} & c_{N^2-2,N^2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1,N^2-2} & c_{1,N^2-3} & \cdots & c_{1,1} & c_{1,N^2-1} \end{Vmatrix} \tag{33}$$

which differs only by an overall sign from $\mathrm{Det}[c_{lj}]$ above, but yields a quasi-block form that generates

$$C_{SU(N)} = \begin{Vmatrix} O & R \\ T & Q \end{Vmatrix}. \tag{34}$$

In this form, R corresponds to the $(N-1)^2$ by $(N-1)^2$ matrix whose determinant is equivalent to $\mathrm{d}V_{SU(N-1)} \cdot \mathrm{d}\alpha_{N^2-1}$, Q is a complicated $2(N-1)$ by $(N-1)^2$ matrix and O is an $(N-1)^2$ by $2(N-1)$ matrix whose elements are all zero. Now the interchange of two columns of an N by N matrix yields a change in sign of the corresponding determinant. But by moving $2(N-1)$ columns at once, the sign of the determinant does not change, and thus one may generate a new matrix which is now block diagonal

$$C_{SU(N)} = \begin{Vmatrix} R & O \\ Q & T \end{Vmatrix}. \tag{35}$$

Thus, the full determinant is just equal to the determinant of the diagonal blocks, one of which is already known. So only the determinant of the $2(N-1)$ by $2(N-1)$ sub-matrix T,

$$\mathrm{Det}[T] = \begin{Vmatrix} c_{2(N-1),N^2-2} & \cdots & c_{2(N-1),(N-1)^2} \\ \cdots & \cdots & \cdots \\ c_{1,N^2-2} & \cdots & c_{1,(N-1)^2} \end{Vmatrix} \tag{36}$$

is needed. Therefore the Haar measure for $SU(N)$ is nothing more than

$$\begin{aligned} \mathrm{d}V_{SU(N)} &= \mathrm{Det}[c_{lj}] \\ &= -\mathrm{Det}[T] * \mathrm{Det}[D]\, \mathrm{d}\alpha_{N^2-1} \cdots \mathrm{d}\alpha_1 \\ &= -\mathrm{Det}[T] *\ \mathrm{d}V_{SU(N-1)}\, \mathrm{d}\alpha_{N^2-1}\, \mathrm{d}\alpha_{2(N-1)} \cdots \mathrm{d}\alpha_1. \end{aligned} \tag{37}$$

This is determined up to normalization (explained in detail in appendix B). Integration over the N^2-1 parameter space yields the following group volume formula (for $N \geqslant 2$),

$$V_{SU(N)} = \int \cdots \int_V \mathrm{d}V_{SU(N)} = \Omega_N * \int \cdots \int_{V'} \mathrm{d}V_{SU(N)} \tag{38}$$

where

$$\begin{aligned} \Omega_N &= 2^{N-2} * \Omega_{(N-1)} * N \\ &= 2^{N-2} * 2^{N-3} \cdots 2^0 * 1 * 2 \cdots (N-1) * N \\ &= 2^{\epsilon} N! \end{aligned} \tag{39}$$

and

$$\begin{aligned} \epsilon &= \sum_{2 \leqslant l \leqslant N} (N-l) \\ &= (N-2) + (N-3) + (N-4) + \cdots + (N-(N-1)) + (N-N) \\ &= (N-2) + (N-3) + (N-4) + \cdots + 1 + 0 \\ &= \tfrac{1}{2}(N-2)(N-1). \end{aligned} \tag{40}$$

6. Example: $SU(5)$ parametrization, Haar measure and group volume

Let $U \in SU(5)$. Following the methodology developed in the previous section, the Euler angle parametrization for $SU(5)$ can be seen to be given by

$$
\begin{aligned}
U &= \prod_{5\geqslant m\geqslant 2}\left(\prod_{2\leqslant k\leqslant m} A(k, j(m))\right) \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{22}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{23}}\,\mathrm{e}^{\mathrm{i}\lambda_{24}\alpha_{24}} \\
A(k, j(m)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}}\,\mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}} \\
j(m) &= \begin{cases} 0 & m=5 \\ \sum_{0\leqslant l\leqslant 5-m-1} 2(m+l) & m\neq 5. \end{cases}
\end{aligned}
\tag{41}
$$

Expansion yields

$$
\begin{aligned}
U &= \left(\prod_{2\leqslant k\leqslant 5} A(k, j(5))\right)\left(\prod_{2\leqslant k\leqslant 4} A(k, j(4))\right)\left(\prod_{2\leqslant k\leqslant 3} A(k, j(3))\right)\left(\prod_{2\leqslant k\leqslant 2} A(k, j(2))\right) \\
&\quad\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{22}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{23}}\,\mathrm{e}^{\mathrm{i}\lambda_{24}\alpha_{24}} \\
&= A(2, j(5))A(3, j(5))A(4, j(5))A(5, j(5))A(2, j(4))A(3, j(4))A(4, j(4)) \\
&\quad\times A(2, j(3))A(3, j(3))A(2, j(2))\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{22}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{23}}\,\mathrm{e}^{\mathrm{i}\lambda_{24}\alpha_{24}}.
\end{aligned}
\tag{42}
$$

The $j(m)$ values are

$$
\begin{aligned}
j(5) &= 0 \\
j(4) &= \sum_{0\leqslant l\leqslant 5-4-1} 2(m+l) = 2m = 8 \\
j(3) &= \sum_{0\leqslant l\leqslant 5-3-1} 2(m+l) = \sum_{0\leqslant l\leqslant 1} 2(m+l) = 2m + 2(m+1) = 14 \\
j(2) &= \sum_{0\leqslant l\leqslant 5-2-1} 2(m+l) = \sum_{0\leqslant l\leqslant 2} 2(m+l) = 2m + 2(m+1) + 2(m+2) = 18
\end{aligned}
\tag{43}
$$

and the $A(k, j(m))$ components are

$$
\begin{aligned}
A(2, j(5)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_2} \\
A(3, j(5)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_4} \\
A(4, j(5)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_5}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_6} \\
A(5, j(5)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_8} \\
A(2, j(4)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{1+8}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{2+8}} = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{10}} \\
A(3, j(4)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{3+8}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{4+8}} = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{11}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{12}} \\
A(4, j(4)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{5+8}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{6+8}} = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{13}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{14}} \\
A(2, j(3)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{1+14}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{2+14}} = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{15}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{16}} \\
A(3, j(3)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{3+14}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{4+14}} = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{17}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{18}} \\
A(2, j(2)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{1+18}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{2+18}} = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{19}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{20}}.
\end{aligned}
\tag{44}
$$

Thus

$$
\begin{aligned}
U &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_5}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_6}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_8}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{10}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{11}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{12}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{13}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{14}} \\
&\quad\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{15}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{16}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{17}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{18}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{19}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{20}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{21}}\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{22}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{23}}\,\mathrm{e}^{\mathrm{i}\lambda_{24}\alpha_{24}}
\end{aligned}
\tag{45}
$$

with a differential volume element of

$$
\begin{aligned}
\mathrm{d}V_{SU(5)} &= \mathrm{Det}[T]\,\mathrm{d}V_{SU(4)}\,\mathrm{d}\alpha_{24}\,\mathrm{d}\alpha_8\cdots\mathrm{d}\alpha_1 \\
&= \mathrm{Det}[T]\cos(\alpha_{12})^3\cos(\alpha_{14})\cos(\alpha_{18})\sin(2\alpha_{10})\sin(\alpha_{12}) \\
&\quad\times \sin(\alpha_{14})^5\sin(2\alpha_{16})\sin(\alpha_{18})^3\sin(2\alpha_{20})\,\mathrm{d}\alpha_{24}\cdots\mathrm{d}\alpha_1
\end{aligned}
\tag{46}
$$

and where Det$[T]$ is an 8 by 8 matrix composed of the following elements:

$$\begin{aligned} c_{lj} &= \frac{-\mathrm{i}}{2}\,\mathrm{Tr}[\lambda_j \cdot M_l] \\ &= \frac{-\mathrm{i}}{2}\,\mathrm{Tr}\left[\lambda_j \cdot \left(\frac{\partial u}{\partial \alpha_l} u^{-1}\right)\right] \qquad (1 \leqslant l \leqslant 8, 16 \leqslant j \leqslant 23, u = U^T) \end{aligned} \tag{47}$$

which when calculated yields

$$\mathrm{Det}[T] = \cos(\alpha_4)^3 \cos(\alpha_6)^5 \cos(\alpha_8) \sin(2\alpha_2) \sin(\alpha_4) \sin(\alpha_6) \sin(\alpha_8)^7. \tag{48}$$

Explicit calculation of the invariant volume element for $SU(5)$ can be done by using equations (38) and (39) and the material from appendix B. From this one generates

$$\begin{aligned} V_{SU(5)} &= 2^6 * 5! * V(SU(5)/Z_5) \\ &= \frac{\sqrt{5}\pi^{14}}{72} \end{aligned} \tag{49}$$

which is in agreement with the volume obtained by Marinov [3].

It should be noted that in calculating $V_{SU(5)}$ one must use the following ranges of integration from appendix B and expressed in equation (38) as V':

$$\begin{aligned} &0 \leqslant \alpha_{2i-1} \leqslant \pi && 0 \leqslant \alpha_{2i} \leqslant \frac{\pi}{2} && (1 \leqslant i \leqslant 10) \\ &0 \leqslant \alpha_{21} \leqslant \pi && 0 \leqslant \alpha_{22} \leqslant \frac{\pi}{\sqrt{3}} \\ &0 \leqslant \alpha_{23} \leqslant \frac{\pi}{\sqrt{6}} && 0 \leqslant \alpha_{24} \leqslant \frac{\pi}{\sqrt{10}}. \end{aligned} \tag{50}$$

Note that these ranges *do not* cover the group $SU(5)$, but rather $SU(5)/Z_5$. The covering ranges for $SU(5)$, following the work in appendix C, are as follows:

$$\begin{aligned} &0 \leqslant \alpha_1, \alpha_9, \alpha_{15}, \alpha_{19} \leqslant \pi \\ &0 \leqslant \alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}, \alpha_{14}, \alpha_{16}, \alpha_{18}, \alpha_{20} \leqslant \frac{\pi}{2} \\ &0 \leqslant \alpha_3, \alpha_5, \alpha_7, \alpha_{11}, \alpha_{13}, \alpha_{17}, \alpha_{21} \leqslant 2\pi \\ &0 \leqslant \alpha_{22} \leqslant \sqrt{3}\pi \\ &0 \leqslant \alpha_{23} \leqslant 2\sqrt{\frac{2}{3}}\pi \\ &0 \leqslant \alpha_{24} \leqslant \sqrt{\frac{5}{2}}\pi. \end{aligned} \tag{51}$$

7. Generalized differential volume element (Haar measure) for $SU(N)$

Let us now quickly review the differential volume element kernels for the first few $SU(N)$ groups (here rewritten in the order of the parametrization of the group):

$$\mathrm{d}V_{SU(2)} = \sin(2\alpha_2)\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_1 \tag{52}$$

$$\begin{aligned} \mathrm{d}V_{SU(3)} &= \sin(2\alpha_2)\cos(\alpha_4)\sin(\alpha_4)^3 \times \mathrm{d}V_{SU(2)}\,\mathrm{d}\alpha_8\,\mathrm{d}\alpha_4 \cdots \mathrm{d}\alpha_1 \\ &= \sin(2\alpha_2)\cos(\alpha_4)\sin(\alpha_4)^3 \sin(2\alpha_6)\,\mathrm{d}\alpha_8 \cdots \mathrm{d}\alpha_1 \end{aligned} \tag{53}$$

$$\begin{aligned} \mathrm{d}V_{SU(4)} &= \sin(2\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)\sin(\alpha_6)^5 \times \mathrm{d}V_{SU(3)}\,\mathrm{d}\alpha_{15}\,\mathrm{d}\alpha_6 \cdots \mathrm{d}\alpha_1 \\ &= \sin(2\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)\sin(\alpha_6)^5\sin(2\alpha_8)\cos(\alpha_{10}) \\ &\quad \times \sin(\alpha_{10})^3 \sin(2\alpha_{12})\,\mathrm{d}\alpha_{15} \cdots \mathrm{d}\alpha_1 \end{aligned} \tag{54}$$

$$\begin{aligned} dV_{SU(5)} &= \sin(2\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)^5\sin(\alpha_6)\cos(\alpha_8)\sin(\alpha_8)^7 \\ &\quad \times dV_{SU(4)}\, d\alpha_{24}\, d\alpha_8 \cdots d\alpha_1 \\ &= \sin(2\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)^5\sin(\alpha_6) \\ &\quad \times \cos(\alpha_8)\sin(\alpha_8)^7\sin(2\alpha_{10})\cos(\alpha_{12})^3\sin(\alpha_{12})\cos(\alpha_{14})\sin(\alpha_{14})^5 \\ &\quad \times \sin(2\alpha_{16})\cos(\alpha_{18})\sin(\alpha_{18})^3\sin(2\alpha_{20})\, d\alpha_{24}\cdots d\alpha_1. \end{aligned} \tag{55}$$

A pattern is emerging with regard to the trigonometric components of the differential volume element. For example, when one looks at the parametrization of the group and matches the trigonometric function in the differential volume element kernel with its corresponding $e^{i\lambda_2\alpha_m}$ component, one sees $\sin(2\alpha_m)$ terms showing up. It is plain to see then that, in general, the differential volume kernel is made up of trigonometric functions that correspond to group elements that are of the form $e^{i\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}$ and where $j(m)$ is given in equation (19). Therefore, there should be a general expression for the differential volume element kernel for $SU(N)$ that can be written down via inspection of the Euler angle parametrization. We shall now show that this is indeed true, and give the methodology for writing down the differential volume element kernel for $SU(N)$. We will also show that this procedure, after integration, yields Marinov's volume formula for $SU(N)$.

To begin, let us take a look at the differential volume elements and their corresponding parametrizations for $SU(3)$ and $SU(4)$. From the parametrization originally given in [9, 10] for $SU(3)$ we see that the $e^{i\lambda_5\alpha_4}$ term contributes to the $\cos(\alpha_4)\sin(\alpha_4)^3$ term in equation (53). Yet in the parametrization for $SU(4)$, originally given in [1], the first $e^{i\lambda_5\alpha_4}$ term yields *not* $\cos(\alpha_4)\sin(\alpha_4)^3$ but rather $\cos(\alpha_4)^3\sin(\alpha_4)$. It is the second installment of λ_5 in the parametrization for $SU(4)$, which can be seen to occur because of equation (18), which gives us the $\cos(\alpha_4)\sin(\alpha_4)^3$ term. For completeness, we should note that the $\cos(\alpha_6)\sin(\alpha_6)^5$ term in the differential volume element for $SU(4)$ comes from the $e^{i\lambda_{10}\alpha_6}$ term in the parametrization.

When we now look at the differential volume element and corresponding parametrization for $SU(5)$, we see the following relationships:

$$\begin{aligned} e^{i\lambda_5\alpha_4} &\Longrightarrow \cos(\alpha_4)^3\sin(\alpha_4) \\ e^{i\lambda_{10}\alpha_6} &\Longrightarrow \cos(\alpha_6)^5\sin(\alpha_6) \\ e^{i\lambda_{17}\alpha_8} &\Longrightarrow \cos(\alpha_8)\sin(\alpha_8)^7 \\ e^{i\lambda_5\alpha_{12}} &\Longrightarrow \cos(\alpha_{12})^3\sin(\alpha_{12}) \\ e^{i\lambda_{10}\alpha_{14}} &\Longrightarrow \cos(\alpha_{14})\sin(\alpha_{14})^5 \\ e^{i\lambda_5\alpha_{18}} &\Longrightarrow \cos(\alpha_{18})\sin(\alpha_{18})^3. \end{aligned} \tag{56}$$

By combining all these observations from the parametrizations and differential volume element kernels for $SU(2)$ to $SU(5)$ we can see that the following pattern is evident:

$$\begin{aligned} e^{i\lambda_2\alpha_2} &\Longrightarrow \sin(2\alpha_2) && k=2 \\ e^{i\lambda_{(k-1)^2+1}\alpha_{2(k-1)}} &\Longrightarrow \cos(\alpha_{2(k-1)})^{2k-3}\sin(\alpha_{2(k-1)}) && 2<k<N \\ e^{i\lambda_{(N-1)^2+1}\alpha_{2(N-1)}} &\Longrightarrow \cos(\alpha_{2(N-1)})\sin(\alpha_{2(N-1)})^{2N-3} && k=N \end{aligned} \tag{57}$$

relating the $A(k)$ term of the recurrence relation given in equation (18) with the Det$[T]$ term of the differential volume element given in equation (37). Exploiting this recurrence relation we now have a methodology for writing down the kernel for any $SU(N)$ differential volume element. Recalling equation (19) and the $A(k, j(m))$ term, we see that in general

$$\begin{aligned} e^{i\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}} &\Longrightarrow \sin(2\alpha_{2+j(m)}) && k=2 \\ &\Longrightarrow \cos(\alpha_{2(k-1)+j(m)})^{2k-3}\sin(\alpha_{2(k-1)+j(m)}) && 2<k<m \\ &\Longrightarrow \cos(\alpha_{2(m-1)+j(m)})\sin(\alpha_{2(m-1)+j(m)})^{2m-3} && k=m \end{aligned} \tag{58}$$

for $N \geqslant m \geqslant 2$. Using this result, combined with knowledge that only these parameters contribute to the integrated kernel in $\mathrm{d}V_{SU(N)}$, we are able to write the following product relation for the kernel of the Haar measure for $SU(N)$:

$$\mathrm{d}V_{SU(N)} = K_{SU(N)}\,\mathrm{d}\alpha_{N^2-1}\cdots\mathrm{d}\alpha_1 \tag{59}$$

where

$$\begin{aligned}
K_{SU(N)} &= \prod_{N\geqslant m\geqslant 2}\left(\prod_{2\leqslant k\leqslant m}\mathrm{Ker}(k, j(m))\right)\\
\mathrm{Ker}(k, j(m)) &= \begin{cases}\sin(2\alpha_{2+j(m)}) & k=2\\ \cos(\alpha_{2(k-1)+j(m)})^{2k-3}\sin(\alpha_{2(k-1)+j(m)}) & 2<k<m\\ \cos(\alpha_{2(m-1)+j(m)})\sin(\alpha_{2(m-1)+j(m)})^{2m-3} & k=m\end{cases}
\end{aligned} \tag{60}$$

and $j(m)$ is from equation (19).

8. Example: $SU(6)$ Haar measure and volume calculation

As proof of the validity of equation (60) we shall use it to write down the differential volume element for $SU(6)$. Observation of equation (37) tells us that $\mathrm{d}V_{SU(6)}$ is dependent on the differential volume elements of $SU(5)$, $SU(4)$, $SU(3)$ and $SU(2)$. Thus, in the process of writing down $\mathrm{d}V_{SU(6)}$ we will not only confirm the calculated differential volume elements for the previous four $SU(N)$ $(N = 5, 4, 3, 2)$ groups, but we will also be able to give the functional form of $\mathrm{Det}[T]$ in equation (37) for $SU(6)$ without having to formally calculate the 10 by 10 determinant. So, for $N = 6$, equations (60) and (19) yield

$$\begin{aligned}
K_{SU(6)} &= \prod_{6\geqslant m\geqslant 2}\left(\prod_{2\leqslant k\leqslant m}\mathrm{Ker}(k, j(m))\right)\\
\mathrm{Ker}(k, j(m)) &= \begin{cases}\sin(2\alpha_{2+j(m)}) & k=2\\ \cos(\alpha_{2(k-1)+j(m)})^{2k-3}\sin(\alpha_{2(k-1)+j(m)}) & 2<k<m\\ \cos(\alpha_{2(m-1)+j(m)})\sin(\alpha_{2(m-1)+j(m)})^{2m-3} & k=m\end{cases}\\
j(m) &= \begin{cases}0 & m=6\\ \sum_{0\leqslant l\leqslant 6-m-1} 2(m+l) & m\neq 6\end{cases}
\end{aligned} \tag{61}$$

which when expanded gives

$$\begin{aligned}
K_{SU(6)} &= \left(\prod_{2\leqslant k\leqslant 6}\mathrm{Ker}(k, j(6))\right)\left(\prod_{2\leqslant k\leqslant 5}\mathrm{Ker}(k, j(5))\right)\left(\prod_{2\leqslant k\leqslant 4}\mathrm{Ker}(k, j(4))\right)\\
&\quad\times\left(\prod_{2\leqslant k\leqslant 3}\mathrm{Ker}(k, j(3))\right)\left(\prod_{2\leqslant k\leqslant 2}\mathrm{Ker}(k, j(2))\right)\\
&= (\mathrm{Ker}(2, j(6))\mathrm{Ker}(3, j(6))\mathrm{Ker}(4, j(6))\mathrm{Ker}(5, j(6))\mathrm{Ker}(6, j(6)))\\
&\quad\times(\mathrm{Ker}(2, j(5))\mathrm{Ker}(3, j(5))\mathrm{Ker}(4, j(5))\mathrm{Ker}(5, j(5)))\\
&\quad\times(\mathrm{Ker}(2, j(4))\mathrm{Ker}(3, j(4))\mathrm{Ker}(4, j(4)))\\
&\quad\times(\mathrm{Ker}(2, j(3))\mathrm{Ker}(3, j(3)))\\
&\quad\times\mathrm{Ker}(2, j(2)).
\end{aligned} \tag{62}$$

The $j(m)$ values are

$$\begin{aligned}
j(6) &= 0 \\
j(5) &= \sum_{0\leqslant l\leqslant 6-5-1} 2(m+l) = 2m = 10 \\
j(4) &= \sum_{0\leqslant l\leqslant 6-4-1} 2(m+l) = \sum_{0\leqslant l\leqslant 1} 2(m+l) = 2m + 2(m+1) = 18 \\
j(3) &= \sum_{0\leqslant l\leqslant 6-3-1} 2(m+l) = \sum_{0\leqslant l\leqslant 2} 2(m+l) = 2m + 2(m+1) + 2(m+2) = 24 \\
j(2) &= \sum_{0\leqslant l\leqslant 6-2-1} 2(m+l) = \sum_{0\leqslant l\leqslant 3} 2(m+l) \\
&= 2m + 2(m+1) + 2(m+2) + 2(m+3) = 28
\end{aligned} \tag{63}$$

and the $\mathrm{Ker}(k, j(m))$ components are

$$\begin{aligned}
\mathrm{Ker}(2, j(6)) &= \sin(2\alpha_2) \\
\mathrm{Ker}(3, j(6)) &= \cos(\alpha_4)^3 \sin(\alpha_4) \\
\mathrm{Ker}(4, j(6)) &= \cos(\alpha_6)^5 \sin(\alpha_6) \\
\mathrm{Ker}(5, j(6)) &= \cos(\alpha_8)^7 \sin(\alpha_8) \\
\mathrm{Ker}(6, j(6)) &= \cos(\alpha_{10}) \sin(\alpha_{10})^9 \\
\mathrm{Ker}(2, j(5)) &= \sin(2\alpha_{2+10}) = \sin(2\alpha_{12}) \\
\mathrm{Ker}(3, j(5)) &= \cos(\alpha_{4+10})^3 \sin(\alpha_{4+10}) = \cos(\alpha_{14})^3 \sin(\alpha_{14}) \\
\mathrm{Ker}(4, j(5)) &= \cos(\alpha_{6+10})^5 \sin(\alpha_{6+10}) = \cos(\alpha_{16})^5 \sin(\alpha_{16}) \\
\mathrm{Ker}(5, j(5)) &= \cos(\alpha_{8+10}) \sin(\alpha_{8+10})^7 = \cos(\alpha_{18}) \sin(\alpha_{18})^7 \\
\mathrm{Ker}(2, j(4)) &= \sin(2\alpha_{2+18}) = \sin(2\alpha_{20}) \\
\mathrm{Ker}(3, j(4)) &= \cos(\alpha_{4+18})^3 \sin(\alpha_{4+18}) = \cos(\alpha_{22})^3 \sin(\alpha_{22}) \\
\mathrm{Ker}(4, j(4)) &= \cos(\alpha_{6+18}) \sin(\alpha_{6+18})^5 = \cos(\alpha_{24}) \sin(\alpha_{24})^5 \\
\mathrm{Ker}(2, j(3)) &= \sin(2\alpha_{2+24}) = \sin(2\alpha_{26}) \\
\mathrm{Ker}(3, j(3)) &= \cos(\alpha_{4+24}) \sin(\alpha_{4+24})^3 = \cos(\alpha_{28}) \sin(\alpha_{28})^3 \\
\mathrm{Ker}(2, j(2)) &= \sin(2\alpha_{2+28}) = \sin(2\alpha_{30}).
\end{aligned} \tag{64}$$

Thus

$$\begin{aligned}
K_{SU(6)} = {} & \sin(2\alpha_2)\cos(\alpha_4)^3 \sin(\alpha_4)\cos(\alpha_6)^5 \sin(\alpha_6)\cos(\alpha_8)^7 \sin(\alpha_8)\cos(\alpha_{10})\sin(\alpha_{10})^9 \\
& \times \sin(2\alpha_{12})\cos(\alpha_{14})^3 \sin(\alpha_{14})\cos(\alpha_{16})^5 \sin(\alpha_{16})\cos(\alpha_{18})\sin(\alpha_{18})^7 \\
& \times \sin(2\alpha_{20})\cos(\alpha_{22})^3 \sin(\alpha_{22})\cos(\alpha_{24})\sin(\alpha_{24})^5 \\
& \times \sin(2\alpha_{26})\cos(\alpha_{28})\sin(\alpha_{28})^3 \sin(2\alpha_{30})
\end{aligned} \tag{65}$$

and

$$\mathrm{d}V_{SU(6)} = K_{SU(6)}\,\mathrm{d}\alpha_{35}\cdots\mathrm{d}\alpha_1. \tag{66}$$

Comparison of the above kernel with those from equations (52), (53), (54) and (55) confirms that equation (60) does correctly yield the differential volume element for $SU(6)$. As added proof, integration of $K_{SU(6)}$ using equations (38) and (39) combined with the following ranges (the general derivation of which can be found in appendix B),

$$\begin{aligned}
&0 \leqslant \alpha_{2i-1} \leqslant \pi && 0 \leqslant \alpha_{2i} \leqslant \frac{\pi}{2} && (1 \leqslant i \leqslant 15) \\
&0 \leqslant \alpha_{31} \leqslant \pi && 0 \leqslant \alpha_{32} \leqslant \frac{\pi}{\sqrt{3}} && 0 \leqslant \alpha_{33} \leqslant \frac{\pi}{\sqrt{6}} \\
&0 \leqslant \alpha_{34} \leqslant \frac{\pi}{\sqrt{10}} && 0 \leqslant \alpha_{35} \leqslant \frac{\pi}{\sqrt{15}}
\end{aligned} \tag{67}$$

yields

$$\begin{aligned} V_{SU(6)} &= 2^{10} * 6! * V(SU(6)/Z_6) \\ &= \frac{\pi^{20}}{1440\sqrt{3}} \end{aligned} \tag{68}$$

which is in agreement with the volume obtained by Marinov [3].

9. Generalized group volume for $SU(N)$

In looking at equations (38) and (60) we see that under this Euler angle parametrization for $SU(N)$ the calculation of the invariant group volume is simply a matter of successive integrations of sines and cosines, multiplied by some power of π and a normalization constant. Therefore it stands to reason that with the derivation of a generalized form for the differential volume element of $SU(N)$, i.e. the Haar measure, for $SU(N)$, there should be a corresponding generalized form for the volume element for $SU(N)$. It is to this derivation that we now focus our attention.

We begin by noticing that in our Euler angle parametrization we have a total of $N^2 - 1$ parameters, of which the final $N - 1$ are the Cartan subalgebra elements for $SU(N)$, thus leaving $N(N - 1)$ elements evenly split between the $\lambda_{(k-1)^2+1}$ and λ_3 parameters (for $2 \leqslant k \leqslant N$). Rewriting this observation using equation (19) we get

$$\begin{aligned} \frac{N(N-1)}{2} &\Longrightarrow \mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}} \\ \frac{N(N-1)}{2} &\Longrightarrow \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}} \\ N-1 &\Longrightarrow \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{N^2-(N-1)}} \cdots \mathrm{e}^{\mathrm{i}\lambda_{N^2-1}\alpha_{N^2-1}}. \end{aligned} \tag{69}$$

Examination of equations (59) and (60) shows that the $N - 1$ Cartan subalgebra elements and the $N(N-1)/2$ λ_3 elements do not contribute to the integrated kernel, but their corresponding parameters are integrated over. Expanding the general results from appendix B, we see that we can use the following ranges to calculate the group volume for $SU(N)$:

$$0 \leqslant \alpha_{2i-1} \leqslant \pi \qquad 0 \leqslant \alpha_{2i} \leqslant \frac{\pi}{2} \qquad 1 \leqslant i \leqslant \frac{N(N-1)}{2} \tag{70}$$

and

$$0 \leqslant \alpha_{N^2+b} \leqslant \pi\sqrt{\frac{2}{a(a-1)}} \qquad a \equiv b \bmod (N+1) \quad a = 2, \ldots, N. \tag{71}$$

From these ranges, it becomes apparent that each λ_3 element contributes a factor of π to the total integration of the differential volume element over the $N^2 - 1$ parameter space while each of the $N - 1$ Cartan subalgebra elements contributes not only a factor of π but a multiplicative constant as well. Explicitly

$$\int_0^{\pi} \mathrm{d}\alpha_{2i-1} = \pi \qquad 1 \leqslant i \leqslant \frac{N(N-1)}{2} \qquad \Longrightarrow \qquad \pi^{\frac{N(N-1)}{2}} \quad \text{from the } \lambda_3 \text{ components} \tag{72}$$

and

$$\begin{aligned}\int_0^{\pi\sqrt{\frac{2}{a(a-1)}}} \mathrm{d}\alpha_{N^2+b} &= \pi\sqrt{\frac{2}{a(a-1)}} \qquad a \equiv b \bmod (N+1) \quad a = 2,\ldots,N \\ &= \pi\sqrt{\frac{2}{k(k-1)}} \qquad 2 \leqslant k \leqslant N \\ &\Longrightarrow \prod_{2\leqslant k\leqslant N} \pi\sqrt{\frac{2}{k(k-1)}} \quad \text{from the } N-1 \text{ Cartan subalgebra components.}\end{aligned} \tag{73}$$

We now focus our attention on the integration of the differential volume element kernel given in equation (60). Examination of the $\mathrm{Ker}(k, j(m))$ term combined with the previously given ranges yields the following three integrals to be evaluated,

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \sin(2\alpha_{2+j(m)})\,\mathrm{d}\alpha_{2+j(m)} \qquad k=2 \\ &\int_0^{\frac{\pi}{2}} \cos(\alpha_{2(k-1)+j(m)})^{2k-3}\sin(\alpha_{2(k-1)+j(m)}))\,\mathrm{d}\alpha_{2(k-1)+j(m)} \qquad 2<k<m \\ &\int_0^{\frac{\pi}{2}} \cos(\alpha_{2(m-1)+j(m)})\sin(\alpha_{2(m-1)+j(m)})^{2m-3})\,\mathrm{d}\alpha_{2(m-1)+j(m)} \qquad k=m\end{aligned} \tag{74}$$

and where, again, $j(m)$ is from equation (19). The first integral is equal to 1, and the other two can be solved by using the following integral solution from Dwight's 'Tables of Integrals and Other Mathematical Data':

$$\int_0^{\frac{\pi}{2}} \sin^p(x)\cos^q(x)\,\mathrm{d}x = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)} \qquad p+1, q+1 > 0 \tag{75}$$

where $\Gamma(x)$ is the standard Gamma function with the following properties:

$$\Gamma(1) = 1 \qquad \Gamma(x+1) = x\Gamma(x). \tag{76}$$

Thus when $p = 1$ and $q = 2k-3$, we get

$$\begin{aligned}\frac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{2k-3+1}{2}\right)}{2\Gamma\left(\frac{1+2k-3}{2}+1\right)} &= \frac{\Gamma(1)\Gamma(k-1)}{2\Gamma(k)} \\ &= \frac{\Gamma(k-1)}{2(k-1)\Gamma(k-1)} \\ &= \frac{1}{2(k-1)}\end{aligned} \tag{77}$$

and when $p = 2m-3$ and $q = 1$, we get

$$\begin{aligned}\frac{\Gamma\left(\frac{2m-3+1}{2}\right)\Gamma\left(\frac{1+1}{2}\right)}{2\Gamma\left(\frac{2m-3+1}{2}+1\right)} &= \frac{\Gamma(m-1)\Gamma(1)}{2\Gamma(m)} \\ &= \frac{\Gamma(m-1)}{2(m-1)\Gamma(m-1)} \\ &= \frac{1}{2(m-1)}.\end{aligned} \tag{78}$$

From this work we can see that with regards to the integration of the $\mathrm{Ker}(k, j(m))$ terms in equation (60) their contribution to the overall calculated volume element is

$$\int_0^{\frac{\pi}{2}} \prod_{N \geqslant m \geqslant 2} \left(\prod_{2 \leqslant k \leqslant m} \mathrm{Ker}(k, j(m)) \right) \mathrm{d}\alpha_{2i} = \prod_{N \geqslant m \geqslant 2} \left(\prod_{2 \leqslant k \leqslant m} \mathbb{V}(k, m) \right)$$

$$\mathrm{d}\alpha_{2i} \text{ runs from } 1 \leqslant i \leqslant \frac{N(N-1)}{2} \tag{79}$$

and where

$$\mathbb{V}(k, m) = \begin{cases} 1 & k = 2 \\ \dfrac{1}{2(k-1)} & 2 < k \leqslant m. \end{cases} \tag{80}$$

Substituting this result as well as equations (72), (73) and (59) into equation (38) yields

$$\begin{aligned} V_{SU(N)} &= \underset{V}{\int \cdots \int} \mathrm{d}V_{SU(N)} \\ &= \Omega_N * \underset{V'}{\int \cdots \int} K_{SU(N)} \, \mathrm{d}\alpha_{N^2-1} \cdots \mathrm{d}\alpha_1 \\ &= 2^{\frac{(N-1)(N-2)}{2}} N! \pi^{\frac{N(N-1)}{2}} \left(\prod_{2 \leqslant k \leqslant N} \pi \sqrt{\frac{2}{k(k-1)}} \right) \\ &\quad \times \left(\prod_{N \geqslant m \geqslant 2} \left(\prod_{2 \leqslant k \leqslant m} \mathbb{V}(k, m) \right) \right). \end{aligned} \tag{81}$$

This expression can be simplified by observing that

$$\begin{aligned} \prod_{2 \leqslant k \leqslant N} \pi \sqrt{\frac{2}{k(k-1)}} &= 2^{\frac{N-1}{2}} \pi^{(N-1)} \prod_{2 \leqslant k \leqslant N} \sqrt{\frac{1}{k(k-1)}} \\ &= 2^{\frac{N-1}{2}} \pi^{(N-1)} \left(\frac{1}{\sqrt{2*1}} * \frac{1}{\sqrt{3*2}} * \frac{1}{\sqrt{4*3}} * \cdots * \frac{1}{\sqrt{N*(N-1)}} \right) \\ &= 2^{\frac{N-1}{2}} \pi^{(N-1)} \frac{1}{\sqrt{N!(N-1)!}} \end{aligned} \tag{82}$$

and, through the usage of equation (80), that

$$\begin{aligned} \prod_{N \geqslant m \geqslant 2} \left(\prod_{2 \leqslant k \leqslant m} \mathbb{V}(k, m) \right) &= \left(\prod_{2 \leqslant k \leqslant N} \mathbb{V}(k, N) \right) \left(\prod_{2 \leqslant k \leqslant N-1} \mathbb{V}(k, N-1) \right) \\ &\quad \times \cdots \times \left(\prod_{2 \leqslant k \leqslant 4} \mathbb{V}(k, 4) \right) \left(\prod_{2 \leqslant k \leqslant 3} \mathbb{V}(k, 3) \right) \mathbb{V}(2, 2) \\ &= (\mathbb{V}(2, N)\mathbb{V}(3, N) \cdots \mathbb{V}(N, N))(\mathbb{V}(2, N-1)\mathbb{V}(3, N-1) \cdots \\ &\quad \times \mathbb{V}(N-1, N-1)) \times \cdots \times (\mathbb{V}(2, 4)\mathbb{V}(3, 4)\mathbb{V}(4, 4))(\mathbb{V}(2, 3)\mathbb{V}(3, 3))\mathbb{V}(2, 2)) \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{V}(3,N)\mathbb{V}(3,N-1)\cdots\mathbb{V}(3,4)\mathbb{V}(3,3)\times\mathbb{V}(4,N)\mathbb{V}(4,N-1)\cdots\mathbb{V}(4,4)\\
&\quad\times\mathbb{V}(N-1,N)\mathbb{V}(N-1,N-1)\times\mathbb{V}(N,N)\\
&= \left(\frac{1}{4}\right)^{N-2}\times\left(\frac{1}{6}\right)^{N-3}\times\cdots\times\left(\frac{1}{2(N-2)}\right)^{2}\times\left(\frac{1}{2(N-1)}\right)\\
&= \left(\frac{1}{2}\right)^{N-2}\left(\frac{1}{2}\right)^{N-2}\times\left(\frac{1}{2}\right)^{N-3}\left(\frac{1}{3}\right)^{N-3}\times\cdots\times\left(\frac{1}{2}\right)^{2}\left(\frac{1}{(N-2)}\right)^{2}\\
&\quad\times\left(\frac{1}{2}\right)\left(\frac{1}{(N-1)}\right) = 2^{-\frac{(N-2)(N-1)}{2}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right).
\end{aligned}
\tag{83}
$$

By substituting these results back into equation (81), we get

$$
\begin{aligned}
V_{SU(N)} &= 2^{\frac{(N-1)(N-2)}{2}}N!\pi^{\frac{N(N-1)}{2}}2^{\frac{N-1}{2}}\pi^{(N-1)}\frac{1}{\sqrt{N!(N-1)!}}2^{-\frac{(N-2)(N-1)}{2}}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)\\
&= 2^{\frac{N-1}{2}}\pi^{\frac{(N-1)(N+2)}{2}}\sqrt{N}\prod_{k=1}^{N-1}\left(\frac{1}{k!}\right)
\end{aligned}
\tag{84}
$$

which is just Marinov's initial formulation of the volume of $SU(N)$ [2, 3]. This is an important result, for it shows that the overall generalized Euler angle parametrization of $SU(N)$ gives results that are consistent with previously scrutinized work which used a completely different methodology to derive the invariant volume of $SU(N)$.[10]

10. *N* by *N* density matrix parametrization

We now turn out attention to the parametrization of N by N density matrices. We state that one may parametrize any N by N density matrix as (see [18] for more details)

$$\rho = U\rho_d U^{\dagger} \tag{85}$$

where ρ_d is the diagonalized density matrix which corresponds to the eigenvalues of the $(N-1)$-sphere, S_{N-1},

$$
\begin{aligned}
\rho_d &= \begin{pmatrix}
\sin^2(\theta_1)\cdots\sin^2(\theta_{N-1}) & 0 & \dots & 0\\
0 & \cos^2(\theta_1)\cdots\sin^2(\theta_{N-1}) & \dots & 0\\
\dots & \dots & \dots & \dots\\
0 & 0 & \dots & \cos^2(\theta_{N-1})
\end{pmatrix}_{N\times N}\\
&= \begin{pmatrix}
\Lambda_1, & 0 & \dots & 0\\
0 & \Lambda_2 & \dots & 0\\
\dots & \dots & \dots & \dots\\
0 & 0 & \dots & \Lambda_N
\end{pmatrix}_{N\times N}
\end{aligned}
\tag{86}
$$

the range of the θ parameters is given by [17, 18]

$$\cos^{-1}\left(\frac{1}{\sqrt{j+1}}\right) \leqslant \theta_j \leqslant \frac{\pi}{2} \tag{87}$$

[10] Marinov calculated the invariant volumes for all compact simple Lie groups by exploiting the spectral expansion of the Green's function for diffusion on a group manifold [2].

and U is from equation (19). For completeness we should note that $U^\dagger$ is defined to be the conjugate transpose of U, which, through equation (26), can be seen to be equal to

$$U^\dagger = e^{-i\lambda_{N^2-1}\alpha_{N^2-1}}\, e^{-i\lambda_{(N-1)^2-1}\alpha_{N^2-2}} \cdots e^{-i\lambda_3\alpha_{N^2-(N-1)}} \prod_{2\leqslant m\leqslant N}\left(\prod_{m\geqslant k\geqslant 2} A(k, j(m))^\dagger\right)$$
$$A(k, j(m))^\dagger = e^{-i\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}\, e^{-i\lambda_3\alpha_{(2k-3)+j(m)}} \tag{88}$$
$$j(m) = \begin{cases} 0 & m = N \\ \sum_{0\leqslant l\leqslant N-m-1} 2(m+l) & m \neq N. \end{cases}$$

Throughout the rest of the paper, ρ_d will be parametrized by the following set of quantities [1, 16–18]:

$$\begin{aligned}\rho_d &= \frac{1}{N}\mathbb{1}_N + \sum_{2\leqslant n\leqslant N} f(\theta_1, \theta_2, \ldots, \theta_{N-1}) * \lambda_{n^2-1} \\ f(\theta_1, \theta_2, \ldots, \theta_{N-1}) &= \tfrac{1}{2}\mathrm{Tr}[\rho_d \cdot \lambda_{n^2-1}].\end{aligned} \tag{89}$$

For example, for the density matrix of two qubits U is given by equation (3), $U^\dagger$ is

$$\begin{aligned}U^\dagger = {}& e^{-i\lambda_{15}\alpha_{15}}\, e^{-i\lambda_8\alpha_{14}}\, e^{-i\lambda_3\alpha_{13}}\, e^{-i\lambda_2\alpha_{12}}\, e^{-i\lambda_3\alpha_{11}}\, e^{-i\lambda_5\alpha_{10}}\, e^{-i\lambda_3\alpha_9}\, e^{-i\lambda_2\alpha_8} \\ & \times e^{-i\lambda_3\alpha_7}\, e^{-i\lambda_{10}\alpha_6}\, e^{-i\lambda_3\alpha_5}\, e^{-i\lambda_5\alpha_4}\, e^{-i\lambda_3\alpha_3}\, e^{-i\lambda_2\alpha_2}\, e^{-i\lambda_3\alpha_1}\end{aligned} \tag{90}$$

and ρ_d is given by

$$\begin{aligned}\rho_d &= \begin{pmatrix} \sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3) & 0 & 0 & 0 \\ 0 & \cos^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3) & 0 & 0 \\ 0 & 0 & \cos^2(\theta_2)\sin^2(\theta_3) & 0 \\ 0 & 0 & 0 & \cos^2(\theta_3) \end{pmatrix} \\ &= \frac{1}{4}\mathbb{1}_4 + \sum_{2\leqslant n\leqslant 4} f(\theta_1, \theta_2, \theta_3) * \lambda_{n^2-1} \\ &= \frac{1}{4}\mathbb{1}_4 + \frac{1}{2}(-1 + 2w^2)x^2y^2 * \lambda_3 \\ &\quad + \frac{1}{2\sqrt{3}}(-2 + 3x^2)y^2 * \lambda_8 + \frac{1}{2\sqrt{6}}(-3 + 4y^2) * \lambda_{15}\end{aligned} \tag{91}$$

where

$$\begin{aligned} & w^2 = \sin^2(\theta_1) && x^2 = \sin^2(\theta_2) && y^2 = \sin^2(\theta_3) \\ & \frac{\pi}{4} \leqslant \theta_1 \leqslant \frac{\pi}{2} && \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \leqslant \theta_2 \leqslant \frac{\pi}{2} && \frac{\pi}{3} \leqslant \theta_3 \leqslant \frac{\pi}{2}\end{aligned} \tag{92}$$

and the one-quarter normalization of $\mathbb{1}_4$ keeps the trace of ρ_d in this form still unity [1]. Therefore, using equations (19), (85), (88) and (89), one can easily write down any N by N density matrix.

11. Example: Haar measures, group volumes and density matrices for qubit/qutrit, three qubit and two qutrit states

Using the formalism we have now established, it is quite easy to write down the Haar measure and group volume for $SU(8)$ and $SU(9)$, as well as the 6 by 6, 8 and 8 and 9 by 9 dimensional density matrices, that correspond to the qubit/qutrit, three qubit and two qutrit states[11].

[11] The Haar measure and group volume for $SU(6)$ has already been written down and calculated in section 8.

11.1. Case 1: qubit/qutrit states

Here, a qubit interacts with a qutrit (a three-state system), thus yielding a six-dimensional Hilbert space which needs to be parametrized. Using equations (19), (85), (88) and (89) we arrive at the following formula for the density matrix of a qubit/qutrit system:

$$U = \prod_{6\geqslant m\geqslant 2}\left(\prod_{2\leqslant k\leqslant m} A(k, j(m))\right) \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{31}}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{32}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{33}}\,\mathrm{e}^{\mathrm{i}\lambda_{24}\alpha_{34}}\,\mathrm{e}^{\mathrm{i}\lambda_{35}\alpha_{35}}$$
$$A(k, j(m)) = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}}\,\mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}$$
$$U^\dagger = \mathrm{e}^{-\mathrm{i}\lambda_{35}\alpha_{35}}\,\mathrm{e}^{-\mathrm{i}\lambda_{24}\alpha_{34}}\,\mathrm{e}^{-\mathrm{i}\lambda_{15}\alpha_{33}}\,\mathrm{e}^{-\mathrm{i}\lambda_8\alpha_{32}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{31}}\prod_{2\leqslant m\leqslant 6}\left(\prod_{m\geqslant k\geqslant 2} A(k, j(m))^\dagger\right)$$
$$A(k, j(m))^\dagger = \mathrm{e}^{-\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}} \tag{93}$$
$$j(m) = \begin{cases} 0 & m = 6 \\ \sum_{0\leqslant l\leqslant 6-m-1} 2(m+l) & m \neq 6 \end{cases}$$
$$\rho_d = \frac{1}{6}\mathbb{1}_6 + \sum_{2\leqslant n\leqslant 6} f(\theta_1, \theta_2, \ldots, \theta_5) * \lambda_{n^2-1}.$$

Explicitly,

$$\begin{aligned}\rho = {} & \mathrm{e}^{\mathrm{i}\lambda_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_5}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_6}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_8}\\
& \times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{10}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{11}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{12}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{13}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{14}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{15}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{16}}\\
& \times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{17}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{18}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{19}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{20}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{22}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{23}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{24}}\\
& \times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{25}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{26}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{27}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{28}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{29}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{30}}(\rho_d)\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{30}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{29}}\\
& \times \mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{28}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{27}}\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{26}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{25}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{24}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{23}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{22}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{21}}\\
& \times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{20}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{19}}\,\mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{18}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{17}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{16}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{15}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{14}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{13}}\\
& \times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{12}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{11}}\,\mathrm{e}^{-\mathrm{i}\lambda_{26}\alpha_{10}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_8}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_6}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_5}\\
& \times \mathrm{e}^{-\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_1}\end{aligned} \tag{94}$$

where

$$\begin{aligned}\rho_d = {} & \frac{1}{6}\mathbb{1}_6 - \frac{\cos(2\theta_1)\sin(\theta_2)^2\sin(\theta_3)^2\sin(\theta_4)^2\sin(\theta_5)^2}{2} * \lambda_3\\
& - \frac{(1+3\cos(2\theta_2))\sin(\theta_3)^2\sin(\theta_4)^2\sin(\theta_5)^2}{4\sqrt{3}} * \lambda_8\\
& - \frac{(1+2\cos(2\theta_3))\sin(\theta_4)^2\sin(\theta_5)^2}{2\sqrt{6}} * \lambda_{15}\\
& - \frac{(3+5\cos(2\theta_4))\sin(\theta_5)^2}{4\sqrt{10}} * \lambda_{24}\\
& - \frac{(2+3\cos(2\theta_5))}{2\sqrt{15}} * \lambda_{35}.\end{aligned} \tag{95}$$

From appendix B and equation (87) we know that the ranges for the quotient group $SU(6)/Z_6$ and the ρ_d parameters are

$$\begin{aligned} & 0 \leqslant \alpha_{2i-1} \leqslant \pi \qquad 0 \leqslant \alpha_{2i} \leqslant \frac{\pi}{2} \qquad (1 \leqslant i \leqslant 15)\\
& \cos^{-1}\left(\frac{1}{\sqrt{j+1}}\right) \leqslant \theta_j \leqslant \frac{\pi}{2} \qquad (1 \leqslant j \leqslant 5).\end{aligned} \tag{96}$$

And, for completeness, we again note that the Haar measure and group volume for a qubit/qutrit system have already been calculated in equation (68).

11.2. Case 2: three qubit states

Here, three qubits interact, thus yielding an eight-dimensional Hilbert space which needs to be parametrized. Using equations (19), (85), (88) and (89) we arrive at the following formula for the density matrix of the three qubit system:

$$
\begin{aligned}
U &= \prod_{8\geqslant m\geqslant 2}\left(\prod_{2\leqslant k\leqslant m} A(k, j(m))\right) \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{57}}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{58}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{59}}\,\mathrm{e}^{\mathrm{i}\lambda_{24}\alpha_{60}}\,\mathrm{e}^{\mathrm{i}\lambda_{35}\alpha_{61}}\,\mathrm{e}^{\mathrm{i}\lambda_{48}\alpha_{62}}\,\mathrm{e}^{\mathrm{i}\lambda_{63}\alpha_{63}} \\
A(k, j(m)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}}\,\mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}} \\
U^\dagger &= \mathrm{e}^{-\mathrm{i}\lambda_{63}\alpha_{63}}\,\mathrm{e}^{-\mathrm{i}\lambda_{48}\alpha_{62}}\,\mathrm{e}^{-\mathrm{i}\lambda_{35}\alpha_{61}}\,\mathrm{e}^{-i\lambda_{24}\alpha_{60}}\,\mathrm{e}^{-\mathrm{i}\lambda_{15}\alpha_{59}}\,\mathrm{e}^{-\mathrm{i}\lambda_8\alpha_{58}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{57}} \\
&\quad\times \prod_{2\leqslant m\leqslant 8}\left(\prod_{m\geqslant k\geqslant 2} A(k, j(m))^\dagger\right) \qquad (97)\\
A(k, j(m))^\dagger &= \mathrm{e}^{-\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}} \\
j(m) &= \begin{cases} 0 & m = 8 \\ \sum_{0\leqslant l\leqslant 8-m-1} 2(m+l) & m \neq 8 \end{cases} \\
\rho_d &= \frac{1}{8}\mathbb{1}_8 + \sum_{2\leqslant n\leqslant 8} f(\theta_1, \theta_2, \ldots, \theta_7) * \lambda_{n^2-1}.
\end{aligned}
$$

Explicitly,

$$
\begin{aligned}
\rho = {}& \mathrm{e}^{\mathrm{i}\lambda_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_5}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_6} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_8}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{10}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{11}}\,\mathrm{e}^{\mathrm{i}\lambda_{37}\alpha_{12}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{13}}\,\mathrm{e}^{\mathrm{i}\lambda_{50}\alpha_{14}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{15}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{16}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{17}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{18}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{19}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{20}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{22}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{23}}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{24}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{25}}\,\mathrm{e}^{\mathrm{i}\lambda_{37}\alpha_{26}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{27}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{28}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{29}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{30}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{31}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{32}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{33}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{34}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{35}}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{36}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{37}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{38}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{39}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{40}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{41}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{42}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{43}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{44}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{45}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{46}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{47}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{48}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{49}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{50}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{51}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{52}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{53}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{54}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{55}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{56}}(\rho_d)\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{56}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{55}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{54}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{53}} \qquad (98)\\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{52}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{51}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{50}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{49}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{48}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{47}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{46}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{45}}\,\mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{44}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{43}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{42}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{41}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{40}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{39}}\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{38}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{37}}\,\mathrm{e}^{-\mathrm{i}\lambda_{26}\alpha_{36}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{35}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{34}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{33}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{32}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{31}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{30}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{29}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{28}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{27}}\,\mathrm{e}^{-\mathrm{i}\lambda_{37}\alpha_{26}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{25}}\,\mathrm{e}^{-\mathrm{i}\lambda_{26}\alpha_{24}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{23}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{22}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{20}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{19}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{18}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{17}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{16}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{15}}\,\mathrm{e}^{-\mathrm{i}\lambda_{50}\alpha_{14}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{13}}\,\mathrm{e}^{-\mathrm{i}\lambda_{37}\alpha_{12}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{11}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_{26}\alpha_{10}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_8}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_6}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_5} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_1}
\end{aligned}
$$

where

$$\rho_d = \frac{\mathbb{1}_8}{8} - \frac{\cos(2\theta_1)\sin(\theta_2)^2\sin(\theta_3)^2\sin(\theta_4)^2\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2}{2} * \lambda_3$$
$$- \frac{(1+3\cos(2\theta_2))\sin(\theta_3)^2\sin(\theta_4)^2\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2}{4\sqrt{3}} * \lambda_8$$
$$- \frac{(1+2\cos(2\theta_3))\sin(\theta_4)^2\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2}{2\sqrt{6}} * \lambda_{15}$$
$$- \frac{(3+5\cos(2\theta_4))\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2}{4\sqrt{10}} * \lambda_{24}$$
$$- \frac{(2+3\cos(2\theta_5))\sin(\theta_6)^2\sin(\theta_7)^2}{2\sqrt{15}} * \lambda_{35}$$
$$- \frac{(5+7\cos(2\theta_6))\sin(\theta_7)^2}{4\sqrt{21}} * \lambda_{48}$$
$$- \frac{(3+4\cos(2\theta_7))}{4\sqrt{7}} * \lambda_{63}. \tag{99}$$

From appendix B and equation (87) we know that the ranges for the quotient group $SU(8)/Z_8$ and the ρ_d parameters are

$$\begin{aligned} &0 \leqslant \alpha_{2i-1} \leqslant \pi \qquad 0 \leqslant \alpha_{2i} \leqslant \frac{\pi}{2} \qquad (1 \leqslant i \leqslant 28) \\ &\cos^{-1}\left(\frac{1}{\sqrt{j+1}}\right) \leqslant \theta_j \leqslant \frac{\pi}{2} \qquad (1 \leqslant j \leqslant 7) \end{aligned} \tag{100}$$

combined with equations (59) and (60), which yield the Haar measure for $SU(8)$

$$\begin{aligned} \mathrm{d}V_{SU(8)} = {} & \sin(2\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)^5\sin(\alpha_6)\cos(\alpha_8)^7\sin(\alpha_8)\cos(\alpha_{10})^9 \\ & \times \sin(\alpha_{10})\cos(\alpha_{12})^{11}\sin(\alpha_{12})\cos(\alpha_{14})\sin(\alpha_{14})^{13} \\ & \times \sin(2\alpha_{16})\cos(\alpha_{18})^3\sin(\alpha_{18})\cos(\alpha_{20})^5\sin(\alpha_{20})\cos(\alpha_{22})^7\sin(\alpha_{22})\cos(\alpha_{24})^9 \\ & \times \sin(\alpha_{24})\cos(\alpha_{26})\sin(\alpha_{26})^{11} \\ & \times \sin(2\alpha_{28})\cos(\alpha_{30})^3\sin(\alpha_{30})\cos(\alpha_{32})^5\sin(\alpha_{32})\cos(\alpha_{34})^7\sin(\alpha_{34})\cos(\alpha_{36}) \\ & \times \sin(\alpha_{36})^9\sin(2\alpha_{38})\cos(\alpha_{40})^3\sin(\alpha_{40})\cos(\alpha_{42})^5\sin(\alpha_{42})\cos(\alpha_{44})\sin(\alpha_{44})^7 \\ & \times \sin(2\alpha_{46})\cos(\alpha_{48})^3\sin(\alpha_{48})\cos(\alpha_{50})\sin(\alpha_{50})^5 \\ & \times \sin(2\alpha_{52})\cos(\alpha_{54})\sin(\alpha_{54})^3 \\ & \times \sin(2\alpha_{56})\,\mathrm{d}\alpha_{63}\cdots\mathrm{d}\alpha_1 \end{aligned} \tag{101}$$

one can calculate the group volume for the three qubit system:

$$\begin{aligned} V_{SU(8)} &= 2^{\frac{8-1}{2}}\pi^{\frac{(8-1)(8+2)}{2}}\sqrt{8}\prod_{k=1}^{8-1}\left(\frac{1}{k!}\right) \\ &= \frac{\pi^{35}}{391\,910\,400} \end{aligned} \tag{102}$$

which is what one gets with $N = 8$ from equation (84).

11.3. Case 3: two qutrits

Here, two qutrits (two three-state systems) interact, thus yielding a nine-dimensional Hilbert space which needs to be parametrized. Using equations (19), (85), (88) and (89) we arrive at the following formula for the density matrix for two qutrits:

$$
\begin{aligned}
U &= \prod_{9\geqslant m\geqslant 2}\left(\prod_{2\leqslant k\leqslant m} A(k,j(m))\right) \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{73}}\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{74}}\,\mathrm{e}^{\mathrm{i}\lambda_{15}\alpha_{75}}\,\mathrm{e}^{\mathrm{i}\lambda_{24}\alpha_{76}}\,\mathrm{e}^{\mathrm{i}\lambda_{35}\alpha_{77}}\,\mathrm{e}^{\mathrm{i}\lambda_{48}\alpha_{78}}\,\mathrm{e}^{\mathrm{i}\lambda_{63}\alpha_{79}}\,\mathrm{e}^{\mathrm{i}\lambda_{80}\alpha_{80}} \\
A(k,j(m)) &= \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}}\,\mathrm{e}^{\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}} \\
U^\dagger &= \mathrm{e}^{-\mathrm{i}\lambda_{80}\alpha_{80}}\,\mathrm{e}^{-\mathrm{i}\lambda_{63}\alpha_{79}}\,\mathrm{e}^{-\mathrm{i}\lambda_{48}\alpha_{78}}\,\mathrm{e}^{-\mathrm{i}\lambda_{35}\alpha_{77}}\,\mathrm{e}^{-\mathrm{i}\lambda_{24}\alpha_{76}}\,\mathrm{e}^{-\mathrm{i}\lambda_{15}\alpha_{75}}\,\mathrm{e}^{-\mathrm{i}\lambda_8\alpha_{74}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{73}} \\
&\quad\times\prod_{2\leqslant m\leqslant 9}\left(\prod_{m\geqslant k\geqslant 2} A(k,j(m))^\dagger\right) \\
A(k,j(m))^\dagger &= \mathrm{e}^{-\mathrm{i}\lambda_{(k-1)^2+1}\alpha_{2(k-1)+j(m)}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{(2k-3)+j(m)}} \\
j(m) &= \begin{cases} 0 & m=9 \\ \sum\limits_{0\leqslant l\leqslant 9-m-1} 2(m+l) & m\neq 9 \end{cases} \\
\rho_d &= \frac{1}{9}\mathbb{1}_9 + \sum_{2\leqslant n\leqslant 9} f(\theta_1,\theta_2,\ldots,\theta_8)*\lambda_{n^2-1}.
\end{aligned}
\tag{103}
$$

Explicitly,

$$
\begin{aligned}
\rho = {}& \mathrm{e}^{\mathrm{i}\lambda_3\alpha_1}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_3}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_4}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_5}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_6} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_8}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{10}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{11}}\,\mathrm{e}^{\mathrm{i}\lambda_{37}\alpha_{12}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{13}}\,\mathrm{e}^{\mathrm{i}\lambda_{50}\alpha_{14}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{15}}\,\mathrm{e}^{\mathrm{i}\lambda_{65}\alpha_{16}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{17}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{18}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{19}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{20}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{22}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{23}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{24}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{25}}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{26}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{27}}\,\mathrm{e}^{\mathrm{i}\lambda_{37}\alpha_{28}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{29}}\,\mathrm{e}^{\mathrm{i}\lambda_{50}\alpha_{30}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{31}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{32}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{33}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{34}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{35}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{36}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{37}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{38}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{39}}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{40}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{41}}\,\mathrm{e}^{\mathrm{i}\lambda_{37}\alpha_{42}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{43}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{44}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{45}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{46}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{47}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{48}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{49}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{50}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{51}}\,\mathrm{e}^{\mathrm{i}\lambda_{26}\alpha_{52}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{53}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{54}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{55}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{56}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{57}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{58}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{59}}\,\mathrm{e}^{\mathrm{i}\lambda_{17}\alpha_{60}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{61}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{62}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{63}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{64}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{65}}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\alpha_{66}} \\
&\times \mathrm{e}^{\mathrm{i}\lambda_3\alpha_{67}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{68}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{69}}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{70}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{71}}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_{72}} \\
&\times (\rho_d)\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{72}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{71}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{70}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{69}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{68}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{67}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{66}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{65}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{64}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{63}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{62}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{61}}\,\mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{60}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{59}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{58}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{57}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{56}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{55}}\,\mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{54}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{53}}\,\mathrm{e}^{-\mathrm{i}\lambda_{26}\alpha_{52}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{51}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{50}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{49}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{48}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{47}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{46}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{45}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{44}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{43}}\,\mathrm{e}^{-\mathrm{i}\lambda_{37}\alpha_{42}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{41}}\,\mathrm{e}^{-\mathrm{i}\lambda_{26}\alpha_{40}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{39}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{38}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{37}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{36}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{35}}\,\mathrm{e}^{-\mathrm{i}\lambda_5\alpha_{34}}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_{33}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_2\alpha_{32}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{31}}\,\mathrm{e}^{-\mathrm{i}\lambda_{50}\alpha_{30}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{29}}\,\mathrm{e}^{-\mathrm{i}\lambda_{37}\alpha_{28}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{27}} \\
&\times \mathrm{e}^{-\mathrm{i}\lambda_{26}\alpha_{26}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{25}}\,\mathrm{e}^{-\mathrm{i}\lambda_{17}\alpha_{22}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{21}}\,\mathrm{e}^{-\mathrm{i}\lambda_{10}\alpha_{20}}\,\mathrm{e}^{-\mathrm{i}\lambda_3\alpha_{19}}
\end{aligned}
\tag{104}
$$

$$\times\, e^{-i\lambda_5\alpha_{18}}\, e^{-i\lambda_3\alpha_{17}}\, e^{-i\lambda_{65}\alpha_{16}}\, e^{-i\lambda_3\alpha_{15}}\, e^{-i\lambda_{50}\alpha_{14}}\, e^{-i\lambda_3\alpha_{13}}$$
$$\times\, e^{-i\lambda_{37}\alpha_{12}}\, e^{-i\lambda_3\alpha_{11}}\, e^{-i\lambda_{26}\alpha_{10}}\, e^{-i\lambda_3\alpha_9}\, e^{-i\lambda_{17}\alpha_8}\, e^{-i\lambda_3\alpha_7}$$
$$\times\, e^{-i\lambda_{10}\alpha_6}\, e^{-i\lambda_3\alpha_5}\, e^{-i\lambda_5\alpha_4}\, e^{-i\lambda_3\alpha_3}\, e^{-i\lambda_2\alpha_2}\, e^{-i\lambda_3\alpha_1}$$

where

$$\begin{aligned}\rho_d = \frac{\mathbb{1}_9}{9} &- \frac{\cos(2\theta_1)\sin(\theta_2)^2\sin(\theta_3)^2\sin(\theta_4)^2\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2\sin(\theta_8)^2}{2} * \lambda_3 \\ &- \frac{(1+3\cos(2\theta_2))\sin(\theta_3)^2\sin(\theta_4)^2\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2\sin(\theta_8)^2}{4\sqrt{3}} * \lambda_8 \\ &- \frac{(1+2\cos(2\theta_3))\sin(\theta_4)^2\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2\sin(\theta_8)^2}{2\sqrt{6}} * \lambda_{15} \\ &- \frac{(3+5\cos(2\theta_4))\sin(\theta_5)^2\sin(\theta_6)^2\sin(\theta_7)^2\sin(\theta_8)^2}{4\sqrt{10}} * \lambda_{24} \\ &- \frac{(2+3\cos(2\theta_5))\sin(\theta_6)^2\sin(\theta_7)^2\sin(\theta_8)^2}{2\sqrt{15}} * \lambda_{35} \\ &- \frac{(5+7\cos(2\theta_6))\sin(\theta_7)^2\sin(\theta_8)^2}{4\sqrt{21}} * \lambda_{48} \\ &- \frac{(3+4\cos(2\theta_7))\sin(\theta_8)^2}{4\sqrt{7}} * \lambda_{63} \\ &- \frac{\left(\frac{7}{12} - \frac{3\cos(2\theta_8)}{4}\right)}{2} * \lambda_{80}. \end{aligned} \tag{105}$$

From appendix B and equation (87) we know that the ranges for the quotient group $SU(9)/Z_9$ and the ρ_d parameters are

$$\begin{aligned} &0 \leqslant \alpha_{2i-1} \leqslant \pi \qquad 0 \leqslant \alpha_{2i} \leqslant \frac{\pi}{2} \qquad (1 \leqslant i \leqslant 36) \\ &\cos^{-1}\left(\frac{1}{\sqrt{j+1}}\right) \leqslant \theta_j \leqslant \frac{\pi}{2} \qquad (1 \leqslant j \leqslant 8) \end{aligned} \tag{106}$$

combined with equations (59) and (60), which yield the Haar measure for $SU(9)$

$$\begin{aligned} dV_{SU(9)} = {} & \sin(2\alpha_2)\cos(\alpha_4)^3\sin(\alpha_4)\cos(\alpha_6)^5\sin(\alpha_6)\cos(\alpha_8)^7\sin(\alpha_8)\cos(\alpha_{10})^9 \\ & \times \sin(\alpha_{10})\cos(\alpha_{12})^{11}\sin(\alpha_{12})\cos(\alpha_{14})^{13}\sin(\alpha_{14})\cos(\alpha_{16})\sin(\alpha_{16})^{15} \\ & \times \sin(2\alpha_{18})\cos(\alpha_{20})^3\sin(\alpha_{20})\cos(\alpha_{22})^5\sin(\alpha_{22})\cos(\alpha_{24})^7\sin(\alpha_{24})\cos(\alpha_{26})^9 \\ & \times \sin(\alpha_{26})\cos(\alpha_{28})^{11}\sin(\alpha_{28})\cos(\alpha_{30})\sin(\alpha_{30})^{13} \\ & \times \sin(2\alpha_{32})\cos(\alpha_{34})^3\sin(\alpha_{34})\cos(\alpha_{36})^5\sin(\alpha_{36})\cos(\alpha_{38})^7\sin(\alpha_{38})\cos(\alpha_{40})^9 \\ & \times \sin(\alpha_{40})\cos(\alpha_{42})\sin(\alpha_{42})^{11} \\ & \times \sin(2\alpha_{44})\cos(\alpha_{46})^3\sin(\alpha_{46})\cos(\alpha_{48})^5\sin(\alpha_{48})\cos(\alpha_{50})^7\sin(\alpha_{50})\cos(\alpha_{52}) \\ & \times \sin(\alpha_{52})^9\sin(2\alpha_{54})\cos(\alpha_{56})^3\sin(\alpha_{56})\cos(\alpha_{58})^5\sin(\alpha_{58})\cos(\alpha_{60})\sin(\alpha_{60})^7 \\ & \times \sin(2\alpha_{62})\cos(\alpha_{64})^3\sin(\alpha_{64})\cos(\alpha_{66})\sin(\alpha_{66})^5 \\ & \times \sin(2\alpha_{68})\cos(\alpha_{70})\sin(\alpha_{70})^3 \\ & \times \sin(2\alpha_{72})\, d\alpha_{80}\cdots d\alpha_1 \end{aligned} \tag{107}$$

one can calculate the group volume for the two qutrit system:

$$V_{SU(9)} = 2^{\frac{9-1}{2}} \pi^{\frac{(9-1)(9+2)}{2}} \sqrt{9} \prod_{k=1}^{9-1} \left(\frac{1}{k!}\right)$$
$$= \frac{\pi^{44}}{105\,345\,515\,520\,000}. \tag{108}$$

which is what one gets with $N = 9$ from equation (84).

12. Conclusions/comments

The aim of this paper has been to show an explicit Euler angle parametrization for $SU(N)$ as well as the Hilbert space for all N by N density matrices[12]. This parametrization also yields a general form for the Haar measure for $SU(N)$ as well as confirms Marinov's initial group volume formula for $SU(N)$. It should be noted that such a parametrization could be very useful in numerous areas of study, most notably numerical calculations concerning entanglement and other quantum information parameters[13]. This parametrization may also allow for an in-depth analysis of the convex sets, sub-sets and overall set boundaries of separable and entangled qubit, qutrit and N-trit systems without having to make any initial restrictions as to the type of parametrization and density matrix in question.

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Appendix A. Diagram formalism for $SU(N)$ parametrization

The invocation of 'removal of the redundancies' in order to achieve the necessary $N^2 - 1$ number of parameters in our $SU(N)$ parametrization seems to be an arbitrary addition to an otherwise rigorous mathematical discussion. This assumed arbitrariness is due, in part, to an underlying construction that is the basis for the formulation of the Euler angle parametrization that we have given. The goal here is to describe this construction and to alleviate any remaining sense of arbitrariness in the development of the parametrization[14].

To begin, let us take a two-component column matrix $\mathbf{w}$ given by

$$\mathbf{w} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{A1}$$

We wish to find the minimal number of operations that rotates, or 'mixes', the two components. Since $\mathbf{w}$ is a 2×1 column matrix, our operations must be 2×2 square matrices in order to generate a 2×1 column matrix as output. These 2×2 square matrices are going to be the

[12] This is but one parametrization that can be realized for a unitary group. For example, Dr Vilenkin and Dr Klimyk in *Representation of Lie Groups and Special Functions* [19] have an extensive treatment of the unitary and orthogonal groups using a similar treatment as was used here.

[13] This has already occurred for the $SU(4)$ case; see, for example, *Preprints* quant-ph\0203088 and quant-ph\0207181.

[14] Using diagrams to develop a representation of a group is a standard tool in developing group parametrizations. An excellent example of this is Dr Vilenkin and Dr Klimyk *Representation of Lie Groups and Special Functions* section 10.5 [19].

elements of $SU(2)$. Therefore, we should look at the well-known Euler angle parametrization of $SU(2)$ to find these operations.

Recall that $SU(2)$ is the covering group of $SO(3)$, the group of rotations of three-dimensional Euclidean space, with the following representation:

> Every rotation $R \in SO(3)$ can be parametrized by an axis of rotation $\hat{n}$ and the angle θ of rotation about the axis: $R = (\hat{n}, \theta)$. The axis requires two angles, (α, β), for its specification, so three parameters are needed to specify a general rotation: $SO(3)$ is a three-parameter group [8].

Now, any arbitrary rotation R in Euclidean space can be accomplished in three steps, known as Euler rotations, which are characterized by three angles, known as Euler angles. When one wants an arbitrary rotation in (x, y, z) space, one then does the following:

1. rotate through an angle α about the z-axis,
2. rotate through an angle β about the new y-axis[15],
3. rotate through an angle γ about the new z-axis.

The rotation matrix R describing these operations is thus given by

$$R(\alpha, \beta, \gamma) = R_{z_{\text{new}}}(\gamma) R_{y_{\text{new}}}(\beta) R_z(\alpha) \tag{A2}$$

where

$$R_z(\alpha) = \begin{pmatrix} \cos[\alpha] & \sin[\alpha] & 0 \\ -\sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_{y_{\text{new}}}(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\beta] & \sin[\beta] \\ 0 & -\sin[\beta] & \cos[\beta] \end{pmatrix}$$

$$R_{z_{\text{new}}}(\gamma) = \begin{pmatrix} \cos[\alpha] & \sin[\alpha] & 0 \\ -\sin[\alpha] & \cos[\alpha] & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A3}$$

Unfortunately at this point we have to make a distinction between body-fixed (the new axes) and space-fixed (the original axes) coordinates. For a general rotation, we would like to represent $R(\alpha, \beta, \gamma)$ in terms of space-fixed axes [8, 20] and not body-fixed axes[16]. This is can be achieved in two steps. First one makes the following definitions [21]:

$$R_{y_{\text{new}}}(\beta) = R_z(\alpha) R_y(\beta) (R_z(\alpha))^{-1} \quad \text{and} \quad R_{z_{\text{new}}}(\gamma) = R_{y_{\text{new}}}(\beta) R_z(\gamma) \left(R_{y_{\text{new}}}(\beta)\right)^{-1}. \tag{A4}$$

Then, substituting these definitions back into the body-fixed version of $R(\alpha, \beta, \gamma)$ given in equation (A1), we get

$$\begin{aligned} R(\alpha, \beta, \gamma) &= R_{z_{\text{new}}}(\gamma) R_{y_{\text{new}}}(\beta) R_z(\alpha) \\ &= R_{y_{\text{new}}}(\beta) R_z(\gamma) \left(R_{y_{\text{new}}}(\beta)\right)^{-1} R_{y_{\text{new}}}(\beta) R_z(\alpha) \\ &= R_{y_{\text{new}}}(\beta) R_z(\gamma) R_z(\alpha) \\ &= R_z(\alpha) R_y(\beta) (R_z(\alpha))^{-1} R_z(\gamma) R_z(\alpha) \\ &= R_z(\alpha) R_y(\beta) R_z(\gamma) \end{aligned} \tag{A5}$$

where we define the y-axis rotation as

$$R_y(\beta) = \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \tag{A6}$$

[15] In classical mechanics, this rotation is conventionally done about the new x-axis [20].

[16] This distinction comes from the physical origins of the three-dimensional rotation group: the rotations of a rigid body about a fixed point constitute the group which we now know to be $SO(3)$. The terminology is from the physical distinction between the coordinate frame 'attached' to the rigid body, and the coordinate frame of the 'surrounding' system [20].

and the last step in the derivation was accomplished by noticing that rotations about the same axis commute.

Thus, one generally does a rotation about the z-axis, then about the y-axis, then another rotation about the z-axis, in order to describe the most general rotation of a rigid body. But, if one is rather interested in the 'mixing' of the components of a Euclidean 3-vector describing a point in the rigid body (from the point of view of the space-fixed axes), then a rotation about the z-axis is equivalent to a 'mixing' between the first and second components of the 3-vector, the rotation about the y-axis is equivalent to a 'mixing' of the first and third components, and the second z-axis rotation, can be seen to be yet another 'mixing' of the first and second components[17]. If we look at the second z-axis rotation as imparting an overall phase, then the first two rotations become the key 'mixing' rotations while the third rotation imparts an overall phase to the components of the 3-vector[18].

In conclusion then, we need the relationship between the rotation matrix $R(\alpha, \beta, \gamma)$ and its $SU(2)$ counterpart in order to accomplish the desired 'mixing' of $\mathbf{w}$. This relationship comes from the following mappings between the $SO(3)$ matrix elements and the $SU(2)$ matrix elements (see, for example, [6–8] for more details):

$$\pm\begin{pmatrix} \cos\left[\frac{\alpha}{2}\right] & \mathrm{i}\sin\left[\frac{\alpha}{2}\right] \\ \mathrm{i}\sin\left[\frac{\alpha}{2}\right] & \cos\left[\frac{\alpha}{2}\right] \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\alpha] & -\sin[\alpha] \\ 0 & \sin[\alpha] & \cos[\alpha] \end{pmatrix}$$

$$\pm\begin{pmatrix} \cos\left[\frac{\beta}{2}\right] & -\sin\left[\frac{\beta}{2}\right] \\ -\sin\left[\frac{\beta}{2}\right] & \cos\left[\frac{\beta}{2}\right] \end{pmatrix} \Longrightarrow \begin{pmatrix} \cos[\beta] & 0 & \sin[\beta] \\ 0 & 1 & 0 \\ -\sin[\beta] & 0 & \cos[\beta] \end{pmatrix} \tag{A7}$$

$$\pm\begin{pmatrix} \mathrm{e}^{\mathrm{i}\frac{\gamma}{2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\frac{\gamma}{2}} \end{pmatrix} \Longrightarrow \begin{pmatrix} \cos[\gamma] & -\sin[\gamma] & 0 \\ \sin[\gamma] & \cos[\gamma] & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From these mappings we see that for a $U \in SU(2)$, an equivalent rotation matrix to $R(\alpha, \beta, \gamma)$ can be written as[19]

$$\begin{aligned} U &= \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,\mathrm{e}^{\mathrm{i}\sigma_2\phi}\,\mathrm{e}^{\mathrm{i}\sigma_3\psi} \\ &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\theta+\psi)}\cos[\phi] & \mathrm{e}^{\mathrm{i}(\theta-\psi)}\sin[\phi] \\ -\mathrm{e}^{\mathrm{i}(-\theta+\psi)}\sin[\phi] & \mathrm{e}^{-\mathrm{i}(\theta+\psi)}\cos[\phi] \end{pmatrix}. \end{aligned} \tag{A8}$$

If we act upon $\mathbf{w}$ with equation (A7) we generate the rotated column matrix $\mathbf{W}$,

$$\mathbf{W} = \begin{pmatrix} a * \mathrm{e}^{\mathrm{i}(\theta+\psi)}\cos[\phi] + b * \mathrm{e}^{\mathrm{i}(\theta-\psi)}\sin[\phi] \\ b * \mathrm{e}^{-\mathrm{i}(\theta+\psi)}\cos[\phi] - a * \mathrm{e}^{\mathrm{i}(-\theta+\psi)}\sin[\phi] \end{pmatrix}. \tag{A9}$$

[17] We will see a similar discussion when we look at elements of $SU(3)$ acting on a three-component column matrix.
[18] The fact that we are having the second z-axis rotation impart nothing more than an overall phase is due in part to an understanding that this geometric approach to the parametrization is not to be taken as as a completely rigorous mathematical derivation, but rather as an aid in visualizing the important group elements of the $SU(N)$ Euler angle parametrization (which when combined with the correspondingly appropriate exponentiated Cartan subalgebra components at the end of the representation, generates both the correct number of parameters for the group, as well as correctly imparts the needed overall phase onto 'mixed' column matrix components). For example, in the $SU(2)$ parametrization case, the third rotation is considered part of the Cartan subalgebra of the parametrization, and can therefore be ignored until later in the parametrization's development.
[19] The equivalence is that $\theta = \frac{\alpha}{2}$, $\phi = \frac{\beta}{2}$ and $\psi = \frac{\gamma}{2}$.

Therefore, following the previous discussion concerning $SO(3)$ rotations, a simplified rotated, or 'mixed' form of **w** can be seen to be given by

$$\begin{aligned}\mathbf{W} &= \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,\mathrm{e}^{\mathrm{i}\sigma_2\phi}\mathbf{w}\\ &= \begin{pmatrix} a * \mathrm{e}^{\mathrm{i}\theta}\cos[\phi] + b * \mathrm{e}^{\mathrm{i}\theta}\sin[\phi] \\ b * \mathrm{e}^{-\mathrm{i}\theta}\cos[\phi] - a * \mathrm{e}^{-\mathrm{i}\theta}\sin[\phi]\end{pmatrix}.\end{aligned} \tag{A10}$$

This reduced $SU(2)$ group operation generating **W** can be pictorially represented by the following diagram:

$$\begin{bmatrix} a \\ b \end{bmatrix} \equiv \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,\mathrm{e}^{\mathrm{i}\sigma_2\phi}. \tag{A11}$$

With this pictorial representation of the group action which 'mixes' column matrix components, we can now graphically express and decompose N-component column matrix 'mixings'.

We can start using this graphical method by looking at a three-component column matrix. Here what we want is the minimal number of operations that will effectively 'mix' the three-component column matrix **v**,

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{A12}$$

Comparing with the previous work, we note that since **w** is a 3×1 column matrix, our operations must be 3×3 square matrices in order to generate the necessary 3×1 column matrix as output. These 3×3 square matrices are going to be elements of $SU(3)$. Therefore, we should look at the Euler angle parametrization of $SU(3)$, initially given in [9, 10] and later in [11], to find these operations.

Before we address the full $SU(3)$ group, the necessary rotations of **v** can be expressed by extending our diagram representation of the $SU(2)$ rotations in the following way:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{then} \quad \left(\begin{bmatrix} a \\ b \end{bmatrix} \text{and} \begin{bmatrix} a \\ \\ c \end{bmatrix}\right) \tag{A13}$$

where

$$\left(\begin{bmatrix} a \\ b \end{bmatrix} \text{and} \begin{bmatrix} a \\ \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ \\ c \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A14}$$

Since we wish to rotate, or 'mix', components a and b and components b and c, we can see that this can be done in stages; first by mixing components a and b, then by mixing a and b and a and c together. Therefore, what is needed to accomplish this scheme is the group element of $SU(3)$ that will rotate components a and c, without rotating b (although an overall phase on b is acceptable). This action is represented by the following $SU(3)$ element:

$$\mathrm{e}^{\mathrm{i}\lambda_5\zeta} = \begin{pmatrix} \cos[\zeta] & 0 & \sin[\zeta] \\ 0 & 1 & 0 \\ -\sin[\zeta] & 0 & \cos[\zeta]\end{pmatrix}. \tag{A15}$$

With equation (A14) and following the methodology from the $SU(2)$ work, we can associate the previously diagrammed $SU(3)$ rotation element with the following group element from $SU(3)$:[20]

$$\begin{bmatrix} a \\ \\ c \end{bmatrix} \equiv \mathrm{e}^{\mathrm{i}\lambda_3\eta}\,\mathrm{e}^{\mathrm{i}\lambda_5\zeta}. \tag{A16}$$

Thus the $SU(3)$ rotations, diagrammed in equation (A12), are seen to be equivalent to the following group action[21]:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ \\ c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \equiv \mathrm{e}^{\mathrm{i}\lambda_3\theta_2}\,\mathrm{e}^{\mathrm{i}\lambda_2\phi_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\eta}\,\mathrm{e}^{\mathrm{i}\lambda_5\zeta}\,\mathrm{e}^{\mathrm{i}\lambda_3\theta}\,\mathrm{e}^{\mathrm{i}\lambda_2\phi}. \tag{A17}$$

The extension of this work to 'mixing' four-component, and thus, n-component column matrices is straightforward. For a four-component column matrix $\mathbf{t}$

$$\mathbf{t} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \tag{A18}$$

we schematically represent the necessary rotations as

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{then} \quad \left(\begin{bmatrix} a \\ b \end{bmatrix} \text{and} \begin{bmatrix} a \\ \\ c \end{bmatrix} \right) \quad \text{then} \quad \left(\begin{bmatrix} a \\ b \end{bmatrix} \text{and} \begin{bmatrix} a \\ \\ c \end{bmatrix} \text{and} \begin{bmatrix} a \\ \\ \\ d \end{bmatrix} \right) \tag{A19}$$

where

$$\left(\begin{bmatrix} a \\ b \end{bmatrix} \text{and} \begin{bmatrix} a \\ \\ c \end{bmatrix} \text{and} \begin{bmatrix} a \\ \\ \\ d \end{bmatrix} \right) = \begin{bmatrix} a \\ \\ \\ d \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} a \\ \\ c \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A20}$$

Following the pattern of the previous work, the rotation between components a and d is achieved by acting upon $\mathbf{t}$ with a 4×4 matrix which is going to be an $SU(4)$ group element

$$\mathrm{e}^{\mathrm{i}\lambda_{10}\chi} = \begin{pmatrix} \cos[\chi] & 0 & 0 & \sin[\chi] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin[\chi] & 0 & 0 & \cos[\chi] \end{pmatrix}. \tag{A21}$$

[20] The reason we have a λ_3 and not a λ_8 in the first group element is beyond the scope of this appendix, but is addressed in the following appendices.

[21] In this representation we are using the general extension of the fact that the 'product' of two rotations R_1 and R_2, denoted by $R_1 R_2$, is the transformation resulting from acting *first* with R_2, *then* with R_1 [13].

Mathematically then, the above 'mixing' scheme can be seen to be given by the following group action[22]:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ \\ c \end{bmatrix} \cdot \begin{bmatrix} a \\ \\ \\ d \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ \\ c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$
$$\equiv \mathrm{e}^{\mathrm{i}\lambda_3\theta_3}\,\mathrm{e}^{\mathrm{i}\lambda_2\phi_3}\,\mathrm{e}^{\mathrm{i}\lambda_3\eta_2}\,\mathrm{e}^{\mathrm{i}\lambda_5\zeta_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\epsilon}\,\mathrm{e}^{\mathrm{i}\lambda_{10}\chi}\,\mathrm{e}^{\mathrm{i}\lambda_3\theta_2}\,\mathrm{e}^{\mathrm{i}\lambda_2\phi_2}\,\mathrm{e}^{\mathrm{i}\lambda_3\eta}\,\mathrm{e}^{\mathrm{i}\lambda_5\zeta}\,\mathrm{e}^{\mathrm{i}\lambda_3\theta}\,\mathrm{e}^{\mathrm{i}\lambda_2\phi}. \quad \text{(A22)}$$

In general then, our graphical representation of 'mixing' column matrix components comes from both Biedenharn's work on angular momentum [13] and from a conceptual extension of the Clebsch–Gordon coefficient construction for spin-$\frac{1}{2}$ particles [21]. Recall that for spin-$\frac{1}{2}$ particles, the only part of the $SU(2)$ rotation operator which mixes different quantum number m values in the Clebsch–Gordon coefficients is the middle rotation; the first and third add nothing but an overall phase. As we have seen that middle rotation corresponds to the σ_2 Pauli matrix for $SU(2)$ and to the $\lambda_{(N-1)^2+1}$ matrix for $SU(N)$, $N > 2$. It is these Lie algebra components that when exponentiated, 'mix' the column matrix components in question. One then only needs to 'attach' (matrix multiply) the appropriate exponentiation of the Cartan subalgebra (σ_3 for $SU(2)$, λ_3 and λ_8 for $SU(3)$, and λ_3, λ_8 and λ_{15} for $SU(4)$) at the end of the group action (given in equation (A10) for $SU(2)$, equation (A16) for $SU(3)$ and equation (A21) for $SU(4)$) to achieve the necessary overall phase for the column matrix components. The generalization of this methodology to higher $SU(N)$ groups should now be quite apparent.

Appendix B. Invariant volume element normalization calculations

Before integrating $\mathrm{d}V_{SU(N)}$ we need some group theory. We begin with a digression concerning the centre of a group [7, 22]. If S is a subset of a group G, then the centralizer $C_G(S)$ of S in G is defined by

$$C(S) \equiv C_G(S) = \{x \in G \mid \text{if } s \in S \text{ then } xs = sx\}. \quad \text{(B1)}$$

For example, if $S = \{y\}$, $C(\mathrm{y})$ will be used instead of $C(\{y\})$. Next, the centralizer of G in G is called the centre of G and is denoted by $Z(G)$ or Z:

$$Z(G) \equiv Z = \{z \in G \mid zx = xz \text{ for all } x \in G\} = C_G(G). \quad \text{(B2)}$$

Another way of writing this is

$$Z(G) = \cap\{C(x) \mid x \in G\} = \{z \mid \text{ if } x \in G \text{ then } z \in C(x)\}. \quad \text{(B3)}$$

In other words, the centre is the set of all elements z that commutes with all other elements in the group. Finally, the commutator $[x, y]$ of two elements x and y of a group G is given by the equation

$$[x, y] = x^{-1}y^{-1}xy. \quad \text{(B4)}$$

Now what we want to find is the number of elements in the centre of $SU(N)$ for $N = 2, 3, 4$ and so on. Begin by defining the following:

$$Z_n = \text{ cyclic group of order } n \cong \mathbb{Z}_n \cong Z(SU(N)). \quad \text{(B5)}$$

[22] See previous footnote.

Therefore, the set of all matrices which comprise the centre of $SU(N)$, $Z(SU(N))$ is congruent to Z_N since we know that if G is a finite linear group over a field F, then the set of matrices of the form $\Sigma c_g g$, where $g \in G$ and $c_g \in F$, forms an algebra (in fact, a ring) [8, 22]. For example, for $SU(2)$ we would have

$$Z_2 = \{x \in SU(2) \mid [x, y] \in Z_1 \text{ for all } y \in SU(2)\} \qquad [x, y] = \omega \mathbb{1}_2 \qquad Z_1 = \{\mathbb{1}_2\}. \tag{B6}$$

This would be the set of all 2 by 2 matrix elements such that the commutator relationship would yield the identity matrix multiplied by some non-zero coefficient. In general, this can be written as

$$Z_N = \{x \in SU(N) \mid [x, y] \in Z_1 \text{ for all } y \in SU(N)\} \qquad Z_1 = \{\mathbb{1}_N\}. \tag{B7}$$

This is similar to the result from [7], which shows that the centre of the general linear group of real matrices, $GL_N(\Re)$, is the group of scalar matrices, that is, those of the from $\omega\mathbb{I}$, where $\mathbb{I}$ is the identity element of the group and ω is some multiplicative constant. For $SU(N)$, $\omega\mathbb{I}$ is an Nth root of unity.

To begin our actual search for the normalization constant for our invariant group element, we first again look at the group $SU(2)$. For this group, every element can be written as

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \tag{B8}$$

where $|a|^2 + |b|^2 = 1$. Again, following [7] we can make the following parametrization:

$$a = y_1 - \mathrm{i}y_2 \qquad b = y_3 - \mathrm{i}y_4 \qquad 1 = y_1^2 + y_2^2 + y_3^2 + y_4^2. \tag{B9}$$

The elements (1, 0, 0, 0) and (−1, 0, 0, 0) are anti-podal points, or polar points if one pictures the group as a three-dimensional unit sphere in a four-dimensional space parametrized by y, and thus comprise the elements for the centre group of $SU(2)$ (i.e. $\pm\mathbb{1}_2$). Therefore, the centre of $SU(2)$, Z_2, comprises two elements, $\pm\mathbb{1}_2$.

Now, in our parametrization, the general $SU(2)$ elements are given by

$$D(\mu, \nu, \xi) = \mathrm{e}^{\mathrm{i}\lambda_3\mu}\,\mathrm{e}^{\mathrm{i}\lambda_2\nu}\,\mathrm{e}^{\mathrm{i}\lambda_3\xi} \qquad \mathrm{d}V_{SU(2)} = \sin(2\nu)\,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{d}\xi \tag{B10}$$

with corresponding ranges

$$0 \leqslant \mu, \xi \leqslant \pi \qquad 0 \leqslant \nu \leqslant \frac{\pi}{2}. \tag{B11}$$

Integrating over the volume element $\mathrm{d}V_{SU(2)}$ with the above ranges yields the volume of the group $SU(2)/Z_2$. In other words, the $SU(2)$ group modulo its centre Z_2. In order to get the full volume of the $SU(2)$ group, all one needs to do is multiply the volume of $SU(2)/Z_2$ by the number of identified centre elements; in this case two.

This process can be extended to the $SU(3)$ and $SU(4)$ parametrizations. For $SU(3)$ [9, 10, 16, 17] (here recast as a component of the $SU(4)$ parametrization derived in [1])

$$SU(3) = \mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_8}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{10}}\,D(\alpha_{11}, \alpha_{12}, \alpha_{13})\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{14}}. \tag{B12}$$

Now, we get an initial factor of 2 from the $D(\alpha_{11}, \alpha_{12}, \alpha_{13})$ component. We shall now prove that we get another factor of 2 from the $\mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{10}}$ component as well.

From the commutation relations of the elements of the Lie algebra of $SU(3)$ (see [9] for details) we see that $\{\lambda_3, \lambda_4, \lambda_5, \lambda_8\}$ form a closed subalgebra $SU(2) \times U(1)$.[23]

$$\begin{aligned} &[\lambda_3, \lambda_4] = \mathrm{i}\lambda_5 && [\lambda_3, \lambda_5] = -\mathrm{i}\lambda_4 && [\lambda_3, \lambda_8] = 0 \\ &[\lambda_4, \lambda_5] = \mathrm{i}(\lambda_3 + \sqrt{3}\lambda_8) && [\lambda_4, \lambda_8] = -\mathrm{i}\sqrt{3}\lambda_5 && [\lambda_5, \lambda_8] = \mathrm{i}\sqrt{3}\lambda_4. \end{aligned} \tag{B13}$$

[23] Georgi [12] has stated that λ_2, λ_5 and λ_7 generate an $SU(2)$ subalgebra of $SU(3)$. This fact can be seen in the commutator relationships between these three λ matrices contained in [1] or in [9].

Observation of the four λ matrices with respect to the Pauli spin matrices of $SU(2)$ shows that λ_4 is the $SU(3)$ analogue of σ_1, λ_5 is the $SU(3)$ analogue of σ_2 and λ_8 is the $SU(3)$ analogue of σ_3:

$$\begin{aligned}
&\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \Longrightarrow \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \\
&\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Longrightarrow \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{B14}$$

Thus one may use either $\{\lambda_3, \lambda_5\}$ or $\{\lambda_3, \lambda_5, \lambda_8\}$ to generate an $SU(2)$ subgroup of $SU(3)$. The volume of this $SU(2)$ subgroup of $SU(3)$ must be equal to the volume of the general $SU(2)$ group, $2\pi^2$. If we demand that any element of the $SU(2)$ subgroup of $SU(3)$ have similar ranges as its $SU(2)$ analogue[24], then a multiplicative factor of 2 is required for the $\mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{10}}$ component[25].

Finally, $SU(3)$ has a Z_3 whose elements have the generic form

$$\begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_1^{-1}\eta_2^{-1} \end{pmatrix} \tag{B15}$$

where

$$\eta_1^3 = \eta_2^3 = 1. \tag{B16}$$

Solving for η_1 and η_2 yields the following elements for Z_3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad -\begin{pmatrix} (-1)^{\frac{1}{3}} & 0 & 0 \\ 0 & (-1)^{\frac{1}{3}} & 0 \\ 0 & 0 & (-1)^{\frac{1}{3}} \end{pmatrix} \qquad \begin{pmatrix} (-1)^{\frac{2}{3}} & 0 & 0 \\ 0 & (-1)^{\frac{2}{3}} & 0 \\ 0 & 0 & (-1)^{\frac{2}{3}} \end{pmatrix} \tag{B17}$$

which are the three cube roots of unity. Combining these $SU(3)$ centre elements, a total of three, with two factors of 2 from the previous discussion, yields a total multiplication factor of 12. The volume of $SU(3)$ is then

$$V_{SU(3)} = 2 * 2 * 3 * V(SU(3)/Z_3) = \sqrt{3}\pi^5 \tag{B18}$$

using the ranges given above for the general $SU(2)$ elements, combined with $0 \leqslant \alpha_{14} \leqslant \frac{\pi}{\sqrt{3}}$. Explicitly

$$0 \leqslant \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13} \leqslant \pi \qquad 0 \leqslant \alpha_8, \alpha_{10}, \alpha_{12} \leqslant \frac{\pi}{2} \qquad 0 \leqslant \alpha_{14} \leqslant \frac{\pi}{\sqrt{3}}. \tag{B19}$$

These are modifications of [9, 10, 16, 17, 23] and take into account the updated Marinov group volume values [3].

[24] This requires a normalization factor of $\frac{1}{\sqrt{3}}$ on the maximal range of λ_8 that is explained by the removal of the Z_3 elements of $SU(3)$.

[25] When calculating this volume element, it is important to remember that the closed subalgebra being used is $SU(2) \otimes U(1)$ and therefore the integrated kernel, be it derived either from $\mathrm{e}^{\mathrm{i}\lambda_3\alpha}\mathrm{e}^{\mathrm{i}\lambda_5\beta}\mathrm{e}^{\mathrm{i}\lambda_3\gamma}$ or $\mathrm{e}^{\mathrm{i}\lambda_3\alpha}\mathrm{e}^{\mathrm{i}\lambda_5\beta}\mathrm{e}^{\mathrm{i}\lambda_8\gamma}$, will require contributions from both the $SU(2)$ and $U(1)$ elements.

For $SU(4)$ the process is similar to that used for $SU(3)$, but now with two $SU(2)$ subgroups to worry about. For a $U \in SU(4)$, the derivation of which can be found in [1], we see that

$$U = e^{i\lambda_3\alpha_1} e^{i\lambda_2\alpha_2} e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4} e^{i\lambda_3\alpha_5} e^{i\lambda_{10}\alpha_6}[SU(3)] e^{i\lambda_{15}\alpha_{15}}. \tag{B20}$$

Here, the two $SU(2)$ subalgebras in $SU(4)$ that we are concerned with are $\{\lambda_3, \lambda_4, \lambda_5, \lambda_8, \lambda_{15}\}$ and $\{\lambda_3, \lambda_9, \lambda_{10}, \lambda_8, \lambda_{15}\}$. Both of these $SU(2) \times U(1) \times U(1)$ subalgebras are represented in the parametrization of $SU(4)$ as $SU(2)$ subgroup elements, $e^{i\lambda_3\alpha_3} e^{i\lambda_5\alpha_4}$ and $e^{i\lambda_3\alpha_5} e^{i\lambda_{10}\alpha_6}$. We can see that λ_{10} is the $SU(4)$ analogue of σ_2,[26] and λ_{15} is the $SU(4)$ analogue to σ_3.[27] The demand that all $SU(2)$ subgroups of $SU(4)$ must have a volume equal to $2\pi^2$ is equivalent to having the parameters of the associated elements of the $SU(2)$ subgroup run through similar ranges as their $SU(2)$ analogues[28]. As with $SU(3)$, this restriction yields an overall multiplicative factor of 4 from these two elements[29]. Recalling that the $SU(3)$ element yields a multiplicative factor of 12, all that remains is to determine the multiplicative factor equivalent to the identification of the $SU(4)$ centre, Z_4.

The elements of the centre of $SU(4)$ are similar in form to the ones from $SU(3)$;

$$\begin{pmatrix} \eta_1 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 \\ 0 & 0 & \eta_3 & 0 \\ 0 & 0 & 0 & \eta_1^{-1}\eta_2^{-1}\eta_3^{-1} \end{pmatrix} \tag{B21}$$

where

$$\eta_1^4 = \eta_2^4 = \eta_3^4 = 1. \tag{B22}$$

Solving yields the four roots of unity $\pm \mathbb{1}_4$ and $\pm\, \mathrm{i}\mathbb{1}_4$, where $\mathbb{1}_4$ is the 4×4 identity matrix. So we can see that Z_4 gives another factor of 4, which, when combined with the factor of 4 from the two $SU(2)$ subgroups, and the factor of 12 from the $SU(3)$ elements, gives a total multiplicative factor of 192.

Thus, when we integrate the $SU(4)$ differential volume element with the ranges given previously for the general $SU(2)$ and $SU(3)$ elements combined with the appropriate range for the λ_{15} component (all combined below)

$$\begin{aligned} &0 \leqslant \alpha_1, \alpha_3, \alpha_5, \alpha_7, \alpha_9, \alpha_{11}, \alpha_{13} \leqslant \pi \\ &0 \leqslant \alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12} \leqslant \frac{\pi}{2} \\ &0 \leqslant \alpha_{14} \leqslant \frac{\pi}{\sqrt{3}} \\ &0 \leqslant \alpha_{15} \leqslant \frac{\pi}{\sqrt{6}} \end{aligned} \tag{B23}$$

we get

$$V_{SU(4)} = 2 * 2 * 2 * 2 * 3 * 4 * V(SU(4)/Z_4) = \frac{\sqrt{2}\pi^9}{3}. \tag{B24}$$

[26] We have already discussed λ_5 in the previous section on $SU(3)$.

[27] It is the $SU(4)$ Cartan subalgebra element.

[28] This requires a normalization factor of $\frac{1}{\sqrt{6}}$ on the maximal range of λ_{15} that is explained by the removal of the Z_4 elements of $SU(4)$.

[29] When calculating these volume elements, it is important to remember that the closed subalgebra being used is $SU(2) \times U(1) \times U(1)$ and therefore, as in the $SU(3)$ case, the integrated kernels will require contributions from appropriate Cartan subalgebra elements. For example, the $e^{i\lambda_3\alpha_3}e^{i\lambda_5\alpha_4}$ component is an $SU(2)$ sub-element of the parametrization of $SU(4)$, but in creating its corresponding $SU(2)$ subgroup volume kernel (see the $SU(3)$ discussion), one must remember that it is an $SU(2) \subset SU(3) \subset SU(4)$ and therefore the kernel only requires contributions from the λ_3 and λ_8 components. On the other hand, the $e^{i\lambda_3\alpha_5}e^{i\lambda_{10}\alpha_6}$ element corresponds to an $SU(2) \subset SU(4)$ and therefore, the volume kernel will require contributions from all three Cartan subalgebra elements of $SU(4)$.

This calculated volume for $SU(4)$, the derivation of which can be found in [1], agrees with that from Marinov [3].

From this work, it is plain to see that, in general, the ranges for the λ_3, λ_2 analogues (recall equation (20)), and the remaining Cartan subalgebra components of the parametrization will take the following general form:

$$\begin{gathered} 0 \leqslant \alpha(\lambda_3) \leqslant \pi \qquad 0 \leqslant \alpha(\lambda_{(k-1)^2+1}) \leqslant \frac{\pi}{2} \\ 0 \leqslant \alpha \text{ (Cartan subalgebra elements)} \leqslant \pi\sqrt{\frac{2}{k(k-1)}} \end{gathered} \tag{B25}$$

for $2 \leqslant k \leqslant N$.[30] Also, it is apparent that the elements of the centre of $SU(N)$ will have the form

$$\begin{pmatrix} \eta_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \eta_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \eta_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \eta_1^{-1}\eta_2^{-1}\eta_3^{-1}\dots\eta_{N-1}^{-1} \end{pmatrix} \tag{B26}$$

where

$$\eta_1^N = \eta_2^N = \eta_3^N = \cdots = \eta_{N-1}^N = 1 \tag{B27}$$

thus yielding N elements in $Z(SU(N))$ and a corresponding multiplicative constant of N in the calculation of the group volume. Observation of the previous calculations of the invariant volume element for $SU(2)$, $SU(3)$ and $SU(4)$ also indicates that the Euler angle parametrization for $SU(N)$ yields $(N-2)SU(2)$ subgroups that require multiplication by 2 in order to satisfy the $2\pi^2$ general $SU(2)$ volume requirement. Therefore, if one defines the ranges given in equation (B25) as V', we see that the invariant volume element for $SU(N)$ can be written as

$$V_{SU(N)} = \Omega_N * \int\cdots\int_{V'} \mathrm{d}V_{SU(N)} \tag{B28}$$

where

$$\Omega_N = 2^{N-2} * N * \Omega_{N-1} \tag{B29}$$

since the differential volume element $\mathrm{d}V_{SU(N)}$, given in (37), shows a reliance on the differential volume element of $SU(N-1)$ and therefore on Ω_{N-1}.

Appendix C. Modified parameter ranges for group covering

In order to be complete, we list the modifications to the ranges given in appendix B that affect a covering of $SU(2)$, $SU(3)$, $SU(4)$ and $SU(N)$, in general, without jeopardizing the calculated group volumes.

To begin, in our parametrization, the general $SU(2)$ elements are given by

$$D(\mu, \nu, \xi) = \mathrm{e}^{\mathrm{i}\lambda_3\mu}\,\mathrm{e}^{\mathrm{i}\lambda_2\nu}\,\mathrm{e}^{\mathrm{i}\lambda_3\xi} \qquad \mathrm{d}V_{SU(2)} = \sin(2\nu)\,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{d}\xi \tag{C1}$$

[30] An observant reader will note that we have counted λ_3 twice in the above ranges. The fact of the matter is that we wanted to distinguish between the first and second λ_3 elements in the fundamental $SU(2)$ parametrization. The first is given as the $\alpha(\lambda_3)$ component while the second is regarded as the $SU(2)$ Cartan subalgebra component. The reason for this distinction will be made clearer in appendix C.

with the corresponding ranges for the volume of $SU(2)/Z_2$ given as

$$0 \leqslant \mu, \xi \leqslant \pi \qquad 0 \leqslant \nu \leqslant \frac{\pi}{2}. \tag{C2}$$

In order to generate a covering of $SU(2)$, the ξ parameter must be modified to take into account the uniqueness of the two central group elements, $\pm\mathbb{1}_2$, under spinor transformations. This modification is straightforward enough; ξ's range is multiplied by the number of central group elements in $SU(2)$. The new ranges are thus

$$0 \leqslant \mu \leqslant \pi \qquad 0 \leqslant \nu \leqslant \frac{\pi}{2} \qquad 0 \leqslant \xi \leqslant 2\pi. \tag{C3}$$

These ranges yield both a covering of $SU(2)$ and the correct group volume for $SU(2)$.[31]

One can also see this by looking at the values of the finite group elements under both sets of ranges. To do this, we first partition $D(\mu, \nu, \xi)$ as

$$D((\mu, \nu), \xi) = (\mathrm{e}^{\mathrm{i}\lambda_3\mu}\,\mathrm{e}^{\mathrm{i}\lambda_2\nu})\,\mathrm{e}^{\mathrm{i}\lambda_3\xi} \tag{C4}$$

where

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\lambda_3\mu} &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\mu} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\mu} \end{pmatrix} \\
\mathrm{e}^{\mathrm{i}\lambda_2\nu} &= \begin{pmatrix} \cos(\nu) & \sin(\nu) \\ -\sin(\nu) & \cos(\nu) \end{pmatrix} \\
\mathrm{e}^{\mathrm{i}\lambda_3\xi} &= \begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\xi} \end{pmatrix}.
\end{aligned} \tag{C5}$$

Then looking at $0 \leqslant \xi \leqslant 2\pi$ first we see the following pattern emerge:

$$\begin{aligned}
&\xi = 0 \Longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \mathbb{1}_2 \\
&\xi = \pi \Longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Longrightarrow -\mathbb{1}_2 \\
&\xi = 2\pi \Longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \mathbb{1}_2 \Longleftarrow \xi = 0. \quad \textbf{Repeat}
\end{aligned} \tag{C6}$$

Next, we look at $0 \leqslant \mu \leqslant \pi$ and $0 \leqslant \nu \leqslant \frac{\pi}{2}$

$$\begin{aligned}
&\mu = 0 \qquad \nu = 0 \Longrightarrow \mathbb{1}_2 \\
&\mu = \pi \qquad \nu = 0 \Longrightarrow -\mathbb{1}_2 \\
&\mu = 0 \qquad \nu = \frac{\pi}{2} \Longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&\mu = \pi \qquad \nu = \frac{\pi}{2} \Longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned} \tag{C7}$$

[31] One may interchange μ and ξ's ranges without altering either the volume calculation or the final orientation of a 2-vector under operation by D. This interchange is beneficial when looking at Euler parametrizations beyond $SU(2)$.

Combining these two results yields the following 12 matrices for $D((\mu,\nu),\xi)$:

$$\begin{aligned}
&D((0,0),0)=\begin{pmatrix}1&0\\0&1\end{pmatrix} && D((\pi,0),0)=-\begin{pmatrix}1&0\\0&1\end{pmatrix}\\
&D((0,0),\pi)=-\begin{pmatrix}1&0\\0&1\end{pmatrix} && D((\pi,0),\pi)=\begin{pmatrix}1&0\\0&1\end{pmatrix}\\
&D((0,0),2\pi)=\begin{pmatrix}1&0\\0&1\end{pmatrix} && D((\pi,0),2\pi)=-\begin{pmatrix}1&0\\0&1\end{pmatrix}\\
&D\left(\left(0,\frac{\pi}{2}\right),0\right)=\begin{pmatrix}0&1\\-1&0\end{pmatrix} && D\left(\left(\pi,\frac{\pi}{2}\right),0\right)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\\
&D\left(\left(0,\frac{\pi}{2}\right),\pi\right)=\begin{pmatrix}0&-1\\1&0\end{pmatrix} && D\left(\left(\pi,\frac{\pi}{2}\right),\pi\right)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\\
&D\left(\left(0,\frac{\pi}{2}\right),2\pi\right)=\begin{pmatrix}0&1\\-1&0\end{pmatrix} && D\left(\left(\pi,\frac{\pi}{2}\right),2\pi\right)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.
\end{aligned} \tag{C8}$$

We can see that in the above 12 matrices we have repeated four fundamental forms, each three times,

$$\begin{aligned}
&\mathbb{1}_2 \Longrightarrow 3 && \begin{pmatrix}0&1\\-1&0\end{pmatrix}\Longrightarrow 3\\
&-\mathbb{1}_2 \Longrightarrow 3 && \begin{pmatrix}0&-1\\1&0\end{pmatrix}\Longrightarrow 3
\end{aligned} \tag{C9}$$

when we have run ξ from 0 to 2π. But, if we remove the repeating $\mathbb{1}_2$ term in the original ξ calculations above—in essence forcing ξ to range from 0 to π instead of 2π—we see that we then get the same four fundamental forms, but now each only repeated twice

$$\begin{aligned}
&\mathbb{1}_2 \Longrightarrow 2 && \begin{pmatrix}0&1\\-1&0\end{pmatrix}\Longrightarrow 2\\
&-\mathbb{1}_2 \Longrightarrow 2 && \begin{pmatrix}0&-1\\1&0\end{pmatrix}\Longrightarrow 2.
\end{aligned} \tag{C10}$$

The greatest common divisor of the above list is, obviously, 2, which not only corresponds to the amount that the range of the ξ parameter was divided by, but also to the multiplicative factor of 2 that is required in the calculation of the invariant volume element when using the quotient group ranges. This may seem trivial, but let us now look at $SU(3)$.

For $SU(3)$, here given as a component of the $SU(4)$ parametrization, we know we have one $SU(2)$ subgroup component (from appendix B) as well as the $SU(2)$ contribution $D(\mu,\nu,\xi)$, here rewritten in terms of the $SU(3)$ parameters α_{11}, α_{12} and α_{13}:

$$SU(3)=\mathrm{e}^{\mathrm{i}\lambda_3\alpha_7}\,\mathrm{e}^{\mathrm{i}\lambda_2\alpha_8}\,\mathrm{e}^{\mathrm{i}\lambda_3\alpha_9}\,\mathrm{e}^{\mathrm{i}\lambda_5\alpha_{10}}D(\alpha_{11},\alpha_{12},\alpha_{13})\,\mathrm{e}^{\mathrm{i}\lambda_8\alpha_{14}}. \tag{C11}$$

Therefore, the ranges of α_9 and α_{13} should be modified just as ξ was done in the previous discussion for $SU(2)$.[32] Remembering the discussion in appendix B concerning the central group of $SU(3)$, we can deduce that α_{14}'s ranges should be multiplied by a factor of 3. This yields the following, corrected, ranges for $SU(3)$[33]:

$$\begin{aligned}
&0\leqslant\alpha_7,\alpha_{11}\leqslant\pi && 0\leqslant\alpha_8,\alpha_{10},\alpha_{12}\leqslant\frac{\pi}{2}\\
&0\leqslant\alpha_9,\alpha_{13}\leqslant 2\pi && 0\leqslant\alpha_{14}\leqslant\sqrt{3}\pi.
\end{aligned} \tag{C12}$$

[32] Recall the previous footnote on the interchange of the first and third component ranges in an $SU(2)$ parametrization.

[33] Earlier representations of these ranges for $SU(3)$, for example in [9, 10, 16, 17, 23], were incorrect in that they failed to take into account the updated $SU(N)$ volume formula in [3].

These ranges yield both a covering of $SU(3)$, as well as the correct group volume for $SU(3)$.

For a $U \in SU(4)$, we have two $SU(2)$ subgroup components

$$U = e^{i\lambda_3\alpha_1}\, e^{i\lambda_2\alpha_2}\, e^{i\lambda_3\alpha_3}\, e^{i\lambda_5\alpha_4}\, e^{i\lambda_3\alpha_5}\, e^{i\lambda_{10}\alpha_6}[SU(3)]\, e^{i\lambda_{15}\alpha_{15}}. \tag{C13}$$

As with the $SU(2)$ subgroup ranges in $SU(3)$, the ranges for α_3 and α_5 each get multiplied by 2 and α_{15}'s ranges get multiplied by 4 (the number of $SU(4)$ central group elements). The remaining ranges are either held the same, or modified in the case of the $SU(3)$ element,

$$\begin{aligned} &0 \leqslant \alpha_1, \alpha_7, \alpha_{11} \leqslant \pi \qquad 0 \leqslant \alpha_2, \alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12} \leqslant \frac{\pi}{2} \\ &0 \leqslant \alpha_3, \alpha_5, \alpha_9, \alpha_{13} \leqslant 2\pi \qquad 0 \leqslant \alpha_{14} \leqslant \sqrt{3}\pi \\ &0 \leqslant \alpha_{15} \leqslant 2\sqrt{\frac{2}{3}}\pi. \end{aligned} \tag{C14}$$

These ranges yield both a covering of $SU(4)$, as well as the correct group volume for $SU(4)$.

In general, we can see that by looking at $SU(N)/Z_N$ not only can we arrive at a parametrization of $SU(N)$ with a logically derivable set of ranges that gives the correct group volume, but we can also show how those ranges can be modified to cover the entire group as well without any arbitrariness in assigning values to the parameters.

References

[1] Tilma T, Byrd M and Sudarshan E C G 2002 A parametrization of bipartite systems based on $SU(4)$ euler angles *Preprint* math-ph\0202002

[2] Marinov M S 1980 Invariant volumes of compact groups *J. Phys. A: Math. Gen.* **13** 3357–66

[3] Marinov M S 1981 Correction to 'Invariant volumes of compact groups' *J. Phys. A: Math. Gen.* **14** 543–4

[4] Preskill J 1998 *Lecture Notes for Physics 229: Quantum Information and Computation* (California Institute of Technology)

[5] Hermann R 1966 *Lie Groups for Physicists* (New York: Benjamin)

[6] Greiner W and Müller B 1989 *Quantum Mechanics: Symmetries* (Berlin: Springer)

[7] Artin M 1991 *Algebra* (Englewood Cliffs, NJ: Prentice-Hall)

[8] Sattinger D H and Weaver O L 1991 *Lie Groups and Algebras with Applications to Physics, Geometry and Mathematics* (Berlin: Springer)

[9] Byrd M 1997 The geometry of SU(3) *Preprint* quant-ph\9708015

[10] Byrd M 1998 Differential geometry on SU(3) with applications to three state systems *J. Math. Phys.* **39** 6125

[11] Cvetič M, Gibbons G W, Lu H and Pope C N 2002 Cohomogeneity one manifolds of Spin(7) and G(2) holonomy *Phys. Rev.* D **65** 106004

[12] Georgi H 1999 *Lie Algebras in Particle Physics* (MA: Perseus Books)

[13] Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics: Theory and Application, in Encyclopedia of Mathematics and its Applications* vol 8 ed Gian-Carlo Rota (Reading, MA: Addison-Wesley)

[14] Herstein I N 1975 *Topics in Algebra* (New York: Wiley)

[15] Murnaghan F D 1938 *The Theory of Group Representations* (Baltimore, MD: Johns Hopkins Press)

[16] Byrd M 1999 Geometric phases for three state systems *Preprint* quant-ph\9902061

[17] Byrd M and Slater P 2001 Bures measures over the spaces of two and three-dimensional matrices *Phys. Lett.* A **283** 152–6

[18] Boya Luis J, Byrd M, Mimms M and Sudarshan E C G 1998 Density matrices and geometric phases for n-state systems *Preprint* quant-ph\9810084

[19] Vilenkin N Ja and Klimyk A U 1993 *Representation of Lie Groups and Special Functions* vol 2 (The Netherlands: Kluwer)

[20] Corben H C and Stehle P 1960 *Classical Mechanics* (MA: Wiley)

[21] Breinig M 2001 *The Wigner-Eckart Theorem* The University of Tennessee Web-Based Quantum Mechanics Course, Spring

[22] Scott W R 1964 *Group Theory* (Englewood cliffs, NJ: Prentice-Hall)

[23] Byrd M and Sudarshan E C G 1998 SU(3) revisited *J. Phys. A: Math. Gen.* **31** 9255–68